

CONCRETE MATHEMATICS

A FOUNDATION FOR COMPUTER SCIENCE

GRAHAM



KNUTH



PATASHNIK

SECOND EDITION



FREE SAMPLE CHAPTER

SHARE WITH OTHERS



This page intentionally left blank

C O N C R E T E

M A T H E M A T I C S

Second Edition

Dedicated to Leonhard Euler (1707–1783)

A Foundation for Computer Science

C O N C R E T E

M A T H E M A T I C S

Second Edition

Ronald L. Graham

AT&T Bell Laboratories

Donald E. Knuth

Stanford University

Oren Patashnik

Center for Communications Research



ADDISON-WESLEY

Upper Saddle River, NJ • Boston • Indianapolis • San Francisco
New York • Toronto • Montréal • London • Munich • Paris • Madrid
Capetown • Sydney • Tokyo • Singapore • Mexico City

Library of Congress Cataloging-in-Publication Data

Graham, Ronald Lewis, 1935-

Concrete mathematics : a foundation for computer science / Ronald
L. Graham, Donald E. Knuth, Oren Patashnik. -- 2nd ed.

xiii,657 p. 24 cm.

Bibliography: p. 604

Includes index.

ISBN 978-0-201-55802-9

1. Mathematics. 2. Computer science-Mathematics. I. Knuth,
Donald Ervin, 1938- . II. Patashnik, Oren, 1954- . III. Title.
QA39.2.G733 1994

510-dc20

93-40325

CIP

Internet page <http://www-cs-faculty.stanford.edu/~knuth/gkp.html>
contains current information about this book and related books.

Electronic version by Mathematical Sciences Publishers (MSP), <http://msp.org>

Copyright © 1994, 1989 by Addison-Wesley Publishing Company, Inc.

All rights reserved. No part of this publication may be reproduced, stored in a
retrieval system, or transmitted, in any form or by any means, electronic, mechan-
ical, photocopying, recording, or otherwise, without the prior written permission
of the publisher. Printed in the United States of America.

ISBN-13 978-0-201-55802-9

ISBN-10 0-201-55802-5

First digital release, August 2015

Preface

“Audience, level, and treatment — a description of such matters is what prefaces are supposed to be about.”

— P. R. Halmos [173]

“People do acquire a little brief authority by equipping themselves with jargon: they can pontificate and air a superficial expertise. But what we should ask of educated mathematicians is not what they can speechify about, nor even what they know about the existing corpus of mathematical knowledge, but rather what can they now do with their learning and whether they can actually solve mathematical problems arising in practice. In short, we look for deeds not words.”

— J. Hammersley [176]

THIS BOOK IS BASED on a course of the same name that has been taught annually at Stanford University since 1970. About fifty students have taken it each year — juniors and seniors, but mostly graduate students — and alumni of these classes have begun to spawn similar courses elsewhere. Thus the time seems ripe to present the material to a wider audience (including sophomores).

It was a dark and stormy decade when Concrete Mathematics was born. Long-held values were constantly being questioned during those turbulent years; college campuses were hotbeds of controversy. The college curriculum itself was challenged, and mathematics did not escape scrutiny. John Hammersley had just written a thought-provoking article “On the enfeeblement of mathematical skills by ‘Modern Mathematics’ and by similar soft intellectual trash in schools and universities” [176]; other worried mathematicians [332] even asked, “Can mathematics be saved?” One of the present authors had embarked on a series of books called *The Art of Computer Programming*, and in writing the first volume he (DEK) had found that there were mathematical tools missing from his repertoire; the mathematics he needed for a thorough, well-grounded understanding of computer programs was quite different from what he’d learned as a mathematics major in college. So he introduced a new course, teaching what he wished somebody had taught him.

The course title “Concrete Mathematics” was originally intended as an antidote to “Abstract Mathematics,” since concrete classical results were rapidly being swept out of the modern mathematical curriculum by a new wave of abstract ideas popularly called the “New Math.” Abstract mathematics is a wonderful subject, and there’s nothing wrong with it: It’s beautiful, general, and useful. But its adherents had become deluded that the rest of mathematics was inferior and no longer worthy of attention. The goal of generalization had become so fashionable that a generation of mathematicians had become unable to relish beauty in the particular, to enjoy the challenge of solving quantitative problems, or to appreciate the value of technique. Abstract mathematics was becoming inbred and losing touch with reality; mathematical education needed a concrete counterweight in order to restore a healthy balance.

When DEK taught Concrete Mathematics at Stanford for the first time, he explained the somewhat strange title by saying that it was his attempt

to teach a math course that was hard instead of soft. He announced that, contrary to the expectations of some of his colleagues, he was *not* going to teach the Theory of Aggregates, nor Stone’s Embedding Theorem, nor even the Stone–Čech compactification. (Several students from the civil engineering department got up and quietly left the room.)

Although Concrete Mathematics began as a reaction against other trends, the main reasons for its existence were positive instead of negative. And as the course continued its popular place in the curriculum, its subject matter “solidified” and proved to be valuable in a variety of new applications. Meanwhile, independent confirmation for the appropriateness of the name came from another direction, when Z. A. Melzak published two volumes entitled *Companion to Concrete Mathematics* [267].

The material of concrete mathematics may seem at first to be a disparate bag of tricks, but practice makes it into a disciplined set of tools. Indeed, the techniques have an underlying unity and a strong appeal for many people. When another one of the authors (RLG) first taught the course in 1979, the students had such fun that they decided to hold a class reunion a year later.

But what exactly is Concrete Mathematics? It is a blend of CONTINUOUS and DISCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using a collection of techniques for solving problems. Once you, the reader, have learned the material in this book, all you will need is a cool head, a large sheet of paper, and fairly decent handwriting in order to evaluate horrendous-looking sums, to solve complex recurrence relations, and to discover subtle patterns in data. You will be so fluent in algebraic techniques that you will often find it easier to obtain exact results than to settle for approximate answers that are valid only in a limiting sense.

The major topics treated in this book include sums, recurrences, elementary number theory, binomial coefficients, generating functions, discrete probability, and asymptotic methods. The emphasis is on manipulative technique rather than on existence theorems or combinatorial reasoning; the goal is for each reader to become as familiar with discrete operations (like the greatest-integer function and finite summation) as a student of calculus is familiar with continuous operations (like the absolute-value function and indefinite integration).

Notice that this list of topics is quite different from what is usually taught nowadays in undergraduate courses entitled “Discrete Mathematics.” Therefore the subject needs a distinctive name, and “Concrete Mathematics” has proved to be as suitable as any other.

The original textbook for Stanford’s course on concrete mathematics was the “Mathematical Preliminaries” section in *The Art of Computer Programming* [207]. But the presentation in those 110 pages is quite terse, so another author (OP) was inspired to draft a lengthy set of supplementary notes. The

“The heart of mathematics consists of concrete examples and concrete problems.”

— P. R. Halmos [172]

“It is downright sinful to teach the abstract before the concrete.”

— Z. A. Melzak [267]

Concrete Mathematics is a bridge to abstract mathematics.

“The advanced reader who skips parts that appear too elementary may miss more than the less advanced reader who skips parts that appear too complex.”

— G. Pólya [297]

(We’re not bold enough to try Distinuous Mathematics.)

present book is an outgrowth of those notes; it is an expansion of, and a more leisurely introduction to, the material of *Mathematical Preliminaries*. Some of the more advanced parts have been omitted; on the other hand, several topics not found there have been included here so that the story will be complete.

The authors have enjoyed putting this book together because the subject began to jell and to take on a life of its own before our eyes; this book almost seemed to write itself. Moreover, the somewhat unconventional approaches we have adopted in several places have seemed to fit together so well, after these years of experience, that we can't help feeling that this book is a kind of manifesto about our favorite way to do mathematics. So we think the book has turned out to be a tale of mathematical beauty and surprise, and we hope that our readers will share at least ϵ of the pleasure we had while writing it.

Since this book was born in a university setting, we have tried to capture the spirit of a contemporary classroom by adopting an informal style. Some people think that mathematics is a serious business that must always be cold and dry; but we think mathematics is fun, and we aren't ashamed to admit the fact. Why should a strict boundary line be drawn between work and play? Concrete mathematics is full of appealing patterns; the manipulations are not always easy, but the answers can be astonishingly attractive. The joys and sorrows of mathematical work are reflected explicitly in this book because they are part of our lives.

Students always know better than their teachers, so we have asked the first students of this material to contribute their frank opinions, as "graffiti" in the margins. Some of these marginal markings are merely corny, some are profound; some of them warn about ambiguities or obscurities, others are typical comments made by wise guys in the back row; some are positive, some are negative, some are zero. But they all are real indications of feelings that should make the text material easier to assimilate. (The inspiration for such marginal notes comes from a student handbook entitled *Approaching Stanford*, where the official university line is counterbalanced by the remarks of outgoing students. For example, Stanford says, "There are a few things you cannot miss in this amorphous shape which is Stanford"; the margin says, "Amorphous . . . what the h*** does that mean? Typical of the pseudo-intellectualism around here." Stanford: "There is no end to the potential of a group of students living together." Graffito: "Stanford dorms are like zoos without a keeper.")

The margins also include direct quotations from famous mathematicians of past generations, giving the actual words in which they announced some of their fundamental discoveries. Somehow it seems appropriate to mix the words of Leibniz, Euler, Gauss, and others with those of the people who will be continuing the work. Mathematics is an ongoing endeavor for people everywhere; many strands are being woven into one rich fabric.

*"... a concrete
life preserver
thrown to students
sinking in a sea of
abstraction."
— W. Gottschalk*

*Math graffiti:
Kilroy wasn't Haar.
Free the group.
Nuke the kernel.
Power to the n.
 $N=1 \Rightarrow P=NP$.*

*I have only a
marginal interest
in this subject.*

*This was the most
enjoyable course
I've ever had. But
it might be nice
to summarize the
material as you
go along.*

This book contains more than 500 exercises, divided into six categories:

- **Warmups** are exercises that EVERY READER should try to do when first reading the material.
- **Basics** are exercises to develop facts that are best learned by trying one's own derivation rather than by reading somebody else's.
- **Homework exercises** are problems intended to deepen an understanding of material in the current chapter.
- **Exam problems** typically involve ideas from two or more chapters simultaneously; they are generally intended for use in take-home exams (not for in-class exams under time pressure).
- **Bonus problems** go beyond what an average student of concrete mathematics is expected to handle while taking a course based on this book; they extend the text in interesting ways.
- **Research problems** may or may not be humanly solvable, but the ones presented here seem to be worth a try (without time pressure).

Answers to all the exercises appear in Appendix A, often with additional information about related results. (Of course, the “answers” to research problems are incomplete; but even in these cases, partial results or hints are given that might prove to be helpful.) Readers are encouraged to look at the answers, especially the answers to the warmup problems, but only AFTER making a serious attempt to solve the problem without peeking.

We have tried in Appendix C to give proper credit to the sources of each exercise, since a great deal of creativity and/or luck often goes into the design of an instructive problem. Mathematicians have unfortunately developed a tradition of borrowing exercises without any acknowledgment; we believe that the opposite tradition, practiced for example by books and magazines about chess (where names, dates, and locations of original chess problems are routinely specified) is far superior. However, we have not been able to pin down the sources of many problems that have become part of the folklore. If any reader knows the origin of an exercise for which our citation is missing or inaccurate, we would be glad to learn the details so that we can correct the omission in subsequent editions of this book.

The typeface used for mathematics throughout this book is a new design by Hermann Zapf [227], commissioned by the American Mathematical Society and developed with the help of a committee that included B. Beeton, R. P. Boas, L. K. Durst, D. E. Knuth, P. Murdock, R. S. Palais, P. Renz, E. Swanson, S. B. Whidden, and W. B. Woolf. The underlying philosophy of Zapf's design is to capture the flavor of mathematics as it might be written by a mathematician with excellent handwriting. A handwritten rather than mechanical style is appropriate because people generally create mathematics with pen, pencil,

*I see:
Concrete mathematics means drilling.*

The homework was tough but I learned a lot. It was worth every hour.

Take-home exams are vital — keep them.

Exams were harder than the homework led me to expect.

Cheaters may pass this course by just copying the answers, but they're only cheating themselves.

Difficult exams don't take into account students who have other classes to prepare for.

I'm unaccustomed to this face.

or chalk. (For example, one of the trademarks of the new design is the symbol for zero, '0', which is slightly pointed at the top because a handwritten zero rarely closes together smoothly when the curve returns to its starting point.) The letters are upright, not italic, so that subscripts, superscripts, and accents are more easily fitted with ordinary symbols. This new type family has been named *AMS Euler*, after the great Swiss mathematician Leonhard Euler (1707–1783) who discovered so much of mathematics as we know it today. The alphabets include Euler Text (Aa Bb Cc through Xx Yy Zz), Euler Fraktur (Œa Æb Ćc through X̣r Ÿ)ŋ Ʒj), and Euler Script Capitals (A B C through X Y Z), as well as Euler Greek (Aα Bβ Γγ through Xχ Ψψ Ωω) and special symbols such as ρ and κ̄. We are especially pleased to be able to inaugurate the Euler family of typefaces in this book, because Leonhard Euler's spirit truly lives on every page: Concrete mathematics is Eulerian mathematics.

Dear prof: Thanks for (1) the puns, (2) the subject matter.

The authors are extremely grateful to Andrei Broder, Ernst Mayr, Andrew Yao, and Frances Yao, who contributed greatly to this book during the years that they taught Concrete Mathematics at Stanford. Furthermore we offer 1024 thanks to the teaching assistants who creatively transcribed what took place in class each year and who helped to design the examination questions; their names are listed in Appendix C. This book, which is essentially a compendium of sixteen years' worth of lecture notes, would have been impossible without their first-rate work.

I don't see how what I've learned will ever help me.

Many other people have helped to make this book a reality. For example, we wish to commend the students at Brown, Columbia, CUNY, Princeton, Rice, and Stanford who contributed the choice graffiti and helped to debug our first drafts. Our contacts at Addison-Wesley were especially efficient and helpful; in particular, we wish to thank our publisher (Peter Gordon), production supervisor (Bette Aaronson), designer (Roy Brown), and copy editor (Lyn Dupré). The National Science Foundation and the Office of Naval Research have given invaluable support. Cheryl Graham was tremendously helpful as we prepared the index. And above all, we wish to thank our wives (Fan, Jill, and Amy) for their patience, support, encouragement, and ideas.

I had a lot of trouble in this class, but I know it sharpened my math skills and my thinking skills.

This second edition features a new Section 5.8, which describes some important ideas that Doron Zeilberger discovered shortly after the first edition went to press. Additional improvements to the first printing can also be found on almost every page.

I would advise the casual student to stay away from this course.

We have tried to produce a perfect book, but we are imperfect authors. Therefore we solicit help in correcting any mistakes that we've made. A reward of \$2.56 will gratefully be conveyed to anyone who is the first to report any error, whether it is mathematical, historical, or typographical.

*Murray Hill, New Jersey
and Stanford, California
May 1988 and October 1993*

—RLG
DEK
OP

A Note on Notation

SOME OF THE SYMBOLISM in this book has not (yet?) become standard. Here is a list of notations that might be unfamiliar to readers who have learned similar material from other books, together with the page numbers where these notations are explained. (See the general index, at the end of the book, for references to more standard notations.)

<i>Notation</i>	<i>Name</i>	<i>Page</i>
$\ln x$	natural logarithm: $\log_e x$	276
$\lg x$	binary logarithm: $\log_2 x$	70
$\log x$	common logarithm: $\log_{10} x$	449
$\lfloor x \rfloor$	floor: $\max\{n \mid n \leq x, \text{ integer } n\}$	67
$\lceil x \rceil$	ceiling: $\min\{n \mid n \geq x, \text{ integer } n\}$	67
$x \bmod y$	remainder: $x - y\lfloor x/y \rfloor$	82
$\{x\}$	fractional part: $x \bmod 1$	70
$\sum f(x) \delta x$	indefinite summation	48
$\sum_a^b f(x) \delta x$	definite summation	49
$x^{\overline{n}}$	falling factorial power: $x!/(x-n)!$	47, 211
$x^{\overline{n}}$	rising factorial power: $\Gamma(x+n)/\Gamma(x)$	48, 211
n_j	subfactorial: $n!/0! - n!/1! + \cdots + (-1)^n n!/n!$	194
$\Re z$	real part: x , if $z = x + iy$	64
$\Im z$	imaginary part: y , if $z = x + iy$	64
H_n	harmonic number: $1/1 + \cdots + 1/n$	29
$H_n^{(x)}$	generalized harmonic number: $1/1^x + \cdots + 1/n^x$	277

If you don't understand what the x denotes at the bottom of this page, try asking your Latin professor instead of your math professor.

$f^{(m)}(z)$	mth derivative of f at z	470
$\left[\begin{matrix} n \\ m \end{matrix} \right]$	Stirling cycle number (the “first kind”)	259
$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$	Stirling subset number (the “second kind”)	258
$\left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle$	Eulerian number	267
$\left\langle\left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle\right\rangle$	Second-order Eulerian number	270
$(a_m \dots a_0)_b$	radix notation for $\sum_{k=0}^m a_k b^k$	11
$K(a_1, \dots, a_n)$	continuant polynomial	302
$F\left(\begin{matrix} a, b \\ c \end{matrix} \middle z\right)$	hypergeometric function	205
$\#A$	cardinality: number of elements in the set A	39
$[z^n] f(z)$	coefficient of z^n in $f(z)$	197
$[\alpha.. \beta]$	closed interval: the set $\{x \mid \alpha \leq x \leq \beta\}$	73
$[m = n]$	1 if $m = n$, otherwise 0^*	24
$[m \setminus n]$	1 if m divides n , otherwise 0^*	102
$[m \setminus\setminus n]$	1 if m exactly divides n , otherwise 0^*	146
$[m \perp n]$	1 if m is relatively prime to n , otherwise 0^*	115

*In general, if S is any statement that can be true or false, the bracketed notation $[S]$ stands for 1 if S is true, 0 otherwise.

Throughout this text, we use single-quote marks (‘...’) to delimit text as it is *written*, double-quote marks (“...”) for a phrase as it is *spoken*. Thus, the string of letters ‘string’ is sometimes called a “string”

An expression of the form ‘ a/bc ’ means the same as ‘ $a/(bc)$ ’. Moreover, $\log x / \log y = (\log x) / (\log y)$ and $2n! = 2(n!)$.

Prestressed concrete mathematics is concrete mathematics that’s preceded by a bewildering list of notations.

Also ‘nonstring’ is a string.

Contents

1	Recurrent Problems	1
1.1	The Tower of Hanoi	1
1.2	Lines in the Plane	4
1.3	The Josephus Problem	8
	Exercises	17
2	Sums	21
2.1	Notation	21
2.2	Sums and Recurrences	25
2.3	Manipulation of Sums	30
2.4	Multiple Sums	34
2.5	General Methods	41
2.6	Finite and Infinite Calculus	47
2.7	Infinite Sums	56
	Exercises	62
3	Integer Functions	67
3.1	Floors and Ceilings	67
3.2	Floor/Ceiling Applications	70
3.3	Floor/Ceiling Recurrences	78
3.4	'mod': The Binary Operation	81
3.5	Floor/Ceiling Sums	86
	Exercises	95
4	Number Theory	102
4.1	Divisibility	102
4.2	Primes	105
4.3	Prime Examples	107
4.4	Factorial Factors	111
4.5	Relative Primality	115
4.6	'mod': The Congruence Relation	123
4.7	Independent Residues	126
4.8	Additional Applications	129
4.9	Phi and Mu	133
	Exercises	144
5	Binomial Coefficients	153
5.1	Basic Identities	153
5.2	Basic Practice	172
5.3	Tricks of the Trade	186

5.4	Generating Functions	196	
5.5	Hypergeometric Functions	204	
5.6	Hypergeometric Transformations	216	
5.7	Partial Hypergeometric Sums	223	
5.8	Mechanical Summation	229	
	Exercises	242	
6	Special Numbers		257
6.1	Stirling Numbers	257	
6.2	Eulerian Numbers	267	
6.3	Harmonic Numbers	272	
6.4	Harmonic Summation	279	
6.5	Bernoulli Numbers	283	
6.6	Fibonacci Numbers	290	
6.7	Continuants	301	
	Exercises	309	
7	Generating Functions		320
7.1	Domino Theory and Change	320	
7.2	Basic Maneuvers	331	
7.3	Solving Recurrences	337	
7.4	Special Generating Functions	350	
7.5	Convolutions	353	
7.6	Exponential Generating Functions	364	
7.7	Dirichlet Generating Functions	370	
	Exercises	371	
8	Discrete Probability		381
8.1	Definitions	381	
8.2	Mean and Variance	387	
8.3	Probability Generating Functions	394	
8.4	Flipping Coins	401	
8.5	Hashing	411	
	Exercises	427	
9	Asymptotics		439
9.1	A Hierarchy	440	
9.2	O Notation	443	
9.3	O Manipulation	450	
9.4	Two Asymptotic Tricks	463	
9.5	Euler's Summation Formula	469	
9.6	Final Summations	476	
	Exercises	489	
A	Answers to Exercises		497
B	Bibliography		604
C	Credits for Exercises		632
	Index		637
	List of Tables		657

This page intentionally left blank

3

Integer Functions

WHOLE NUMBERS constitute the backbone of discrete mathematics, and we often need to convert from fractions or arbitrary real numbers to integers. Our goal in this chapter is to gain familiarity and fluency with such conversions and to learn some of their remarkable properties.

3.1 FLOORS AND CEILINGS

We start by covering the floor (greatest integer) and ceiling (least integer) functions, which are defined for all real x as follows:

$$\begin{aligned}\lfloor x \rfloor &= \text{the greatest integer less than or equal to } x; \\ \lceil x \rceil &= \text{the least integer greater than or equal to } x.\end{aligned}\tag{3.1}$$

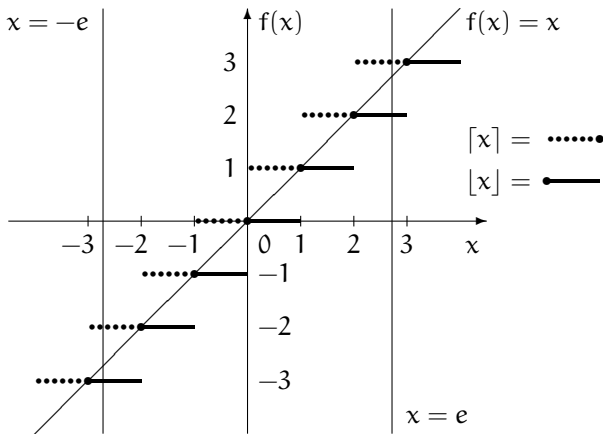
Kenneth E. Iverson introduced this notation, as well as the names “floor” and “ceiling,” early in the 1960s [191, page 12]. He found that typesetters could handle the symbols by shaving the tops and bottoms off of ‘ \lfloor ’ and ‘ \rceil ’. His notation has become sufficiently popular that floor and ceiling brackets can now be used in a technical paper without an explanation of what they mean. Until recently, people had most often been writing ‘ $\lfloor x \rfloor$ ’ for the greatest integer $\leq x$, without a good equivalent for the least integer function. Some authors had even tried to use ‘ $\lceil x \rceil$ ’ — with a predictable lack of success.

)Ouch.(

Besides variations in notation, there are variations in the functions themselves. For example, some pocket calculators have an INT function, defined as $\lfloor x \rfloor$ when x is positive and $\lceil x \rceil$ when x is negative. The designers of these calculators probably wanted their INT function to satisfy the identity $\text{INT}(-x) = -\text{INT}(x)$. But we’ll stick to our floor and ceiling functions, because they have even nicer properties than this.

One good way to become familiar with the floor and ceiling functions is to understand their graphs, which form staircase-like patterns above and

below the line $f(x) = x$:



We see from the graph that, for example,

$$\begin{aligned} [e] &= 2, & [-e] &= -3, \\ [e] &= 3, & [-e] &= -2, \end{aligned}$$

since $e = 2.71828\dots$

By staring at this illustration we can observe several facts about floors and ceilings. First, since the floor function lies on or below the diagonal line $f(x) = x$, we have $[x] \leq x$; similarly $[x] \geq x$. (This, of course, is quite obvious from the definition.) The two functions are equal precisely at the integer points:

$$[x] = x \iff x \text{ is an integer} \iff [x] = x.$$

(We use the notation ‘ \iff ’ to mean “if and only if.”) Furthermore, when they differ the ceiling is exactly 1 higher than the floor:

$$[x] - [x] = [x \text{ is not an integer}]. \tag{3.2}$$

Cute. By Iverson’s bracket convention, this is a complete equation.

If we shift the diagonal line down one unit, it lies completely below the floor function, so $x - 1 < [x]$; similarly $x + 1 > [x]$. Combining these observations gives us

$$x - 1 < [x] \leq x \leq [x] < x + 1. \tag{3.3}$$

Finally, the functions are reflections of each other about both axes:

$$[-x] = -[x]; \quad [-x] = -[x]. \tag{3.4}$$

Thus each is easily expressible in terms of the other. This fact helps to explain why the ceiling function once had no notation of its own. But we see ceilings often enough to warrant giving them special symbols, just as we have adopted special notations for rising powers as well as falling powers. Mathematicians have long had both sine and cosine, tangent and cotangent, secant and cosecant, max and min; now we also have both floor and ceiling.

Next week we're getting walls.

To actually prove properties about the floor and ceiling functions, rather than just to observe such facts graphically, the following four rules are especially useful:

$$\begin{aligned}
 \lfloor x \rfloor = n &\iff n \leq x < n + 1, & (a) \\
 \lfloor x \rfloor = n &\iff x - 1 < n \leq x, & (b) \\
 \lceil x \rceil = n &\iff n - 1 < x \leq n, & (c) \\
 \lceil x \rceil = n &\iff x \leq n < x + 1. & (d)
 \end{aligned} \tag{3.5}$$

(We assume in all four cases that n is an integer and that x is real.) Rules (a) and (c) are immediate consequences of definition (3.1); rules (b) and (d) are the same but with the inequalities rearranged so that n is in the middle.

It's possible to move an integer term in or out of a floor (or ceiling):

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n, \quad \text{integer } n. \tag{3.6}$$

(Because rule (3.5(a)) says that this assertion is equivalent to the inequalities $\lfloor x \rfloor + n \leq x + n < \lfloor x \rfloor + n + 1$.) But similar operations, like moving out a constant factor, cannot be done in general. For example, we have $\lfloor nx \rfloor \neq n \lfloor x \rfloor$ when $n = 2$ and $x = 1/2$. This means that floor and ceiling brackets are comparatively inflexible. We are usually happy if we can get rid of them or if we can prove anything at all when they are present.

It turns out that there are many situations in which floor and ceiling brackets are redundant, so that we can insert or delete them at will. For example, any inequality between a real and an integer is equivalent to a floor or ceiling inequality between integers:

$$\begin{aligned}
 x < n &\iff \lfloor x \rfloor < n, & (a) \\
 n < x &\iff n < \lceil x \rceil, & (b) \\
 x \leq n &\iff \lceil x \rceil \leq n, & (c) \\
 n \leq x &\iff n \leq \lfloor x \rfloor. & (d)
 \end{aligned} \tag{3.7}$$

These rules are easily proved. For example, if $x < n$ then surely $\lfloor x \rfloor < n$, since $\lfloor x \rfloor \leq x$. Conversely, if $\lfloor x \rfloor < n$ then we must have $x < n$, since $x < \lfloor x \rfloor + 1$ and $\lfloor x \rfloor + 1 \leq n$.

It would be nice if the four rules in (3.7) were as easy to remember as they are to prove. Each inequality without floor or ceiling corresponds to the

same inequality with floor or with ceiling; but we need to think twice before deciding which of the two is appropriate.

The difference between x and $\lfloor x \rfloor$ is called the *fractional part* of x , and it arises often enough in applications to deserve its own notation:

$$\{x\} = x - \lfloor x \rfloor. \quad (3.8)$$

We sometimes call $\lfloor x \rfloor$ the *integer part* of x , since $x = \lfloor x \rfloor + \{x\}$. If a real number x can be written in the form $x = n + \theta$, where n is an integer and $0 \leq \theta < 1$, we can conclude by (3.5(a)) that $n = \lfloor x \rfloor$ and $\theta = \{x\}$.

Identity (3.6) doesn't hold if n is an arbitrary real. But we can deduce that there are only two possibilities for $\lfloor x + y \rfloor$ in general: If we write $x = \lfloor x \rfloor + \{x\}$ and $y = \lfloor y \rfloor + \{y\}$, then we have $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor$. And since $0 \leq \{x\} + \{y\} < 2$, we find that sometimes $\lfloor x + y \rfloor$ is $\lfloor x \rfloor + \lfloor y \rfloor$, otherwise it's $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

3.2 FLOOR/CEILING APPLICATIONS

We've now seen the basic tools for handling floors and ceilings. Let's put them to use, starting with an easy problem: What's $\lceil \lg 35 \rceil$? (Following a convention that many authors have proposed independently, we use 'lg' to denote the base-2 logarithm.) Well, since $2^5 < 35 \leq 2^6$, we can take logs to get $5 < \lg 35 \leq 6$; so relation (3.5(c)) tells us that $\lceil \lg 35 \rceil = 6$.

Note that the number 35 is six bits long when written in radix 2 notation: $35 = (100011)_2$. Is it always true that $\lceil \lg n \rceil$ is the length of n written in binary? Not quite. We also need six bits to write $32 = (100000)_2$. So $\lceil \lg n \rceil$ is the wrong answer to the problem. (It fails only when n is a power of 2, but that's infinitely many failures.) We can find a correct answer by realizing that it takes m bits to write each number n such that $2^{m-1} \leq n < 2^m$; thus (3.5(a)) tells us that $m - 1 = \lfloor \lg n \rfloor$, so $m = \lfloor \lg n \rfloor + 1$. That is, we need $\lfloor \lg n \rfloor + 1$ bits to express n in binary, for all $n > 0$. Alternatively, a similar derivation yields the answer $\lceil \lg(n + 1) \rceil$; this formula holds for $n = 0$ as well, if we're willing to say that it takes zero bits to write $n = 0$ in binary.

Let's look next at expressions with several floors or ceilings. What is $\lceil \lfloor x \rfloor \rceil$? Easy—since $\lfloor x \rfloor$ is an integer, $\lceil \lfloor x \rfloor \rceil$ is just $\lfloor x \rfloor$. So is any other expression with an innermost $\lfloor x \rfloor$ surrounded by any number of floors or ceilings.

Here's a tougher problem: Prove or disprove the assertion

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor, \quad \text{real } x \geq 0. \quad (3.9)$$

Equality obviously holds when x is an integer, because $x = \lfloor x \rfloor$. And there's equality in the special cases $\pi = 3.14159\dots$, $e = 2.71828\dots$, and $\phi = (1 + \sqrt{5})/2 = 1.61803\dots$, because we get $1 = 1$. Our failure to find a counterexample suggests that equality holds in general, so let's try to prove it.

Hmmm. We'd better not write $\{x\}$ for the fractional part when it could be confused with the set containing x as its only element.

The second case occurs if and only if there's a "carry" at the position of the decimal point, when the fractional parts $\{x\}$ and $\{y\}$ are added together.

(Of course π , e , and ϕ are the obvious first real numbers to try, aren't they?)

Skepticism is healthy only to a limited extent. Being skeptical about proofs and programs (particularly your own) will probably keep your grades healthy and your job fairly secure. But applying that much skepticism will probably also keep you shut away working all the time, instead of letting you get out for exercise and relaxation. Too much skepticism is an open invitation to the state of rigor mortis, where you become so worried about being correct and rigorous that you never get anything finished.

—A skeptic

Incidentally, when we're faced with a "prove or disprove," we're usually better off trying first to disprove with a counterexample, for two reasons: A disproof is potentially easier (we need just one counterexample); and nit-picking arouses our creative juices. Even if the given assertion is true, our search for a counterexample often leads us to a proof, as soon as we see why a counterexample is impossible. Besides, it's healthy to be skeptical.

If we try to prove that $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$ with the help of calculus, we might start by decomposing x into its integer and fractional parts $\lfloor x \rfloor + \{x\} = n + \theta$ and then expanding the square root using the binomial theorem: $(n + \theta)^{1/2} = n^{1/2} + n^{-1/2}\theta/2 - n^{-3/2}\theta^2/8 + \dots$. But this approach gets pretty messy.

It's much easier to use the tools we've developed. Here's a possible strategy: Somehow strip off the outer floor and square root of $\lfloor \sqrt{\lfloor x \rfloor} \rfloor$, then remove the inner floor, then add back the outer stuff to get $\lfloor \sqrt{x} \rfloor$. OK. We let $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$ and invoke (3.5(a)), giving $m \leq \sqrt{\lfloor x \rfloor} < m + 1$. That removes the outer floor bracket without losing any information. Squaring, since all three expressions are nonnegative, we have $m^2 \leq \lfloor x \rfloor < (m + 1)^2$. That gets rid of the square root. Next we remove the floor, using (3.7(d)) for the left inequality and (3.7(a)) for the right: $m^2 \leq x < (m + 1)^2$. It's now a simple matter to retrace our steps, taking square roots to get $m \leq \sqrt{x} < m + 1$ and invoking (3.5(a)) to get $m = \lfloor \sqrt{x} \rfloor$. Thus $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = m = \lfloor \sqrt{x} \rfloor$; the assertion is true. Similarly, we can prove that

$$\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil, \quad \text{real } x \geq 0.$$

The proof we just found doesn't rely heavily on the properties of square roots. A closer look shows that we can generalize the ideas and prove much more: Let $f(x)$ be any continuous, monotonically increasing function on an interval of the real numbers, with the property that

$$f(x) = \text{integer} \quad \implies \quad x = \text{integer}.$$

(The symbol ' \implies ' means "implies.") Then we have

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \quad \text{and} \quad \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil, \quad (3.10)$$

(This observation was made by R. J. McEliece when he was an undergrad.)

whenever $f(x)$, $f(\lfloor x \rfloor)$, and $f(\lceil x \rceil)$ are defined. Let's prove this general property for ceilings, since we did floors earlier and since the proof for floors is almost the same. If $x = \lceil x \rceil$, there's nothing to prove. Otherwise $x < \lceil x \rceil$, and $f(x) < f(\lceil x \rceil)$ since f is increasing. Hence $\lceil f(x) \rceil \leq \lceil f(\lceil x \rceil) \rceil$, since $\lceil \cdot \rceil$ is nondecreasing. If $\lceil f(x) \rceil < \lceil f(\lceil x \rceil) \rceil$, there must be a number y such that $x \leq y < \lceil x \rceil$ and $f(y) = \lceil f(x) \rceil$, since f is continuous. This y is an integer, because of f 's special property. But there cannot be an integer strictly between $\lfloor x \rfloor$ and $\lceil x \rceil$. This contradiction implies that we must have $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$.

An important special case of this theorem is worth noting explicitly:

$$\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor \quad \text{and} \quad \left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil, \quad (3.11)$$

if m and n are integers and the denominator n is positive. For example, let $m = 0$; we have $\lfloor \lfloor \lfloor x/10 \rfloor / 10 \rfloor / 10 \rfloor = \lfloor x/1000 \rfloor$. Dividing thrice by 10 and throwing off digits is the same as dividing by 1000 and tossing the remainder.

Let's try now to prove or disprove another statement:

$$\lceil \sqrt{\lfloor x \rfloor} \rceil \stackrel{?}{=} \lceil \sqrt{x} \rceil, \quad \text{real } x \geq 0.$$

This works when $x = \pi$ and $x = e$, but it fails when $x = \phi$; so we know that it isn't true in general.

Before going any further, let's digress a minute to discuss different levels of problems that might appear in books about mathematics:

Level 1. Given an explicit object x and an explicit property $P(x)$, *prove* that $P(x)$ is true. For example, "Prove that $\lfloor \pi \rfloor = 3$." Here the problem involves finding a proof of some purported fact.

Level 2. Given an explicit set X and an explicit property $P(x)$, prove that $P(x)$ is true *for all* $x \in X$. For example, "Prove that $\lfloor x \rfloor \leq x$ for all real x ." Again the problem involves finding a proof, but the proof this time must be general. We're doing algebra, not just arithmetic.

Level 3. Given an explicit set X and an explicit property $P(x)$, *prove or disprove* that $P(x)$ is true for all $x \in X$. For example, "Prove or disprove that $\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil$ for all real $x \geq 0$." Here there's an additional level of uncertainty; the outcome might go either way. This is closer to the real situation a mathematician constantly faces: Assertions that get into books tend to be true, but new things have to be looked at with a jaundiced eye. If the statement is false, our job is to find a counterexample. If the statement is true, we must find a proof as in level 2.

Level 4. Given an explicit set X and an explicit property $P(x)$, find a *necessary and sufficient condition* $Q(x)$ that $P(x)$ is true. For example, "Find a necessary and sufficient condition that $\lfloor x \rfloor \geq \lceil x \rceil$." The problem is to find Q such that $P(x) \iff Q(x)$. Of course, there's always a trivial answer; we can take $Q(x) = P(x)$. But the implied requirement is to find a condition that's as simple as possible. Creativity is required to discover a simple condition that will work. (For example, in this case, " $\lfloor x \rfloor \geq \lceil x \rceil \iff x$ is an integer.") The extra element of discovery needed to find $Q(x)$ makes this sort of problem more difficult, but it's more typical of what mathematicians must do in the "real world." Finally, of course, a proof must be given that $P(x)$ is true if and only if $Q(x)$ is true.

In my other texts "prove or disprove" seems to mean the same as "prove," about 99.44% of the time; but not in this book.

But no simpler.
—A. Einstein

Level 5. Given an explicit set X , find an interesting property $P(x)$ of its elements. Now we're in the scary domain of pure research, where students might think that total chaos reigns. This is real mathematics. Authors of textbooks rarely dare to pose level 5 problems.

End of digression. But let's convert the last question we looked at from level 3 to level 4: What is a necessary and sufficient condition that $\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil$? We have observed that equality holds when $x = 3.142$ but not when $x = 1.618$; further experimentation shows that it fails also when x is between 9 and 10. Oho. Yes. We see that bad cases occur whenever $m^2 < x < m^2 + 1$, since this gives m on the left and $m + 1$ on the right. In all other cases where \sqrt{x} is defined, namely when $x = 0$ or $m^2 + 1 \leq x \leq (m + 1)^2$, we get equality. The following statement is therefore necessary and sufficient for equality: Either x is an integer or $\sqrt{\lfloor x \rfloor}$ isn't.

For our next problem let's consider a handy new notation, suggested by C. A. R. Hoare and Lyle Ramshaw, for intervals of the real line: $[\alpha.. \beta]$ denotes the set of real numbers x such that $\alpha \leq x \leq \beta$. This set is called a *closed interval* because it contains both endpoints α and β . The interval containing neither endpoint, denoted by $(\alpha.. \beta)$, consists of all x such that $\alpha < x < \beta$; this is called an *open interval*. And the intervals $[\alpha.. \beta)$ and $(\alpha.. \beta]$, which contain just one endpoint, are defined similarly and called *half-open*.

How many integers are contained in such intervals? The half-open intervals are easier, so we start with them. In fact half-open intervals are almost always nicer than open or closed intervals. For example, they're additive — we can combine the half-open intervals $[\alpha.. \beta)$ and $[\beta.. \gamma)$ to form the half-open interval $[\alpha.. \gamma)$. This wouldn't work with open intervals because the point β would be excluded, and it could cause problems with closed intervals because β would be included twice.

Back to our problem. The answer is easy if α and β are integers: Then $[\alpha.. \beta)$ contains the $\beta - \alpha$ integers $\alpha, \alpha + 1, \dots, \beta - 1$, assuming that $\alpha \leq \beta$. Similarly $(\alpha.. \beta]$ contains $\beta - \alpha$ integers in such a case. But our problem is harder, because α and β are arbitrary reals. We can convert it to the easier problem, though, since

$$\begin{aligned} \alpha \leq n < \beta & \iff \lceil \alpha \rceil \leq n < \lceil \beta \rceil, \\ \alpha < n \leq \beta & \iff \lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor, \end{aligned}$$

when n is an integer, according to (3.7). The intervals on the right have integer endpoints and contain the same number of integers as those on the left, which have real endpoints. So the interval $[\alpha.. \beta)$ contains exactly $\lceil \beta \rceil - \lceil \alpha \rceil$ integers, and $(\alpha.. \beta]$ contains $\lfloor \beta \rfloor - \lfloor \alpha \rfloor$. This is a case where we actually want to introduce floor or ceiling brackets, instead of getting rid of them.

Home of the
Toledo Mudhens.

(Or, by pessimists,
half-closed.)

By the way, there's a mnemonic for remembering which case uses floors and which uses ceilings: Half-open intervals that include the left endpoint but not the right (such as $0 \leq \theta < 1$) are slightly more common than those that include the right endpoint but not the left; and floors are slightly more common than ceilings. So by Murphy's Law, the correct rule is the opposite of what we'd expect — ceilings for $[\alpha.. \beta]$ and floors for $(\alpha.. \beta]$.

Similar analyses show that the closed interval $[\alpha.. \beta]$ contains exactly $\lfloor \beta \rfloor - \lceil \alpha \rceil + 1$ integers and that the open interval $(\alpha.. \beta)$ contains $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$; but we place the additional restriction $\alpha \neq \beta$ on the latter so that the formula won't ever embarrass us by claiming that an empty interval $(\alpha.. \alpha)$ contains a total of -1 integers. To summarize, we've deduced the following facts:

interval	integers contained	restrictions	
$[\alpha.. \beta]$	$\lfloor \beta \rfloor - \lceil \alpha \rceil + 1$	$\alpha \leq \beta,$	
$[\alpha.. \beta)$	$\lfloor \beta \rfloor - \lceil \alpha \rceil$	$\alpha \leq \beta,$	(3.12)
$(\alpha.. \beta]$	$\lfloor \beta \rfloor - \lfloor \alpha \rfloor$	$\alpha \leq \beta,$	
$(\alpha.. \beta)$	$\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$	$\alpha < \beta.$	

Now here's a problem we can't refuse. The Concrete Math Club has a casino (open only to purchasers of this book) in which there's a roulette wheel with one thousand slots, numbered 1 to 1000. If the number n that comes up on a spin is divisible by the floor of its cube root, that is, if

$$\lfloor \sqrt[3]{n} \rfloor \setminus n,$$

then it's a winner and the house pays us \$5; otherwise it's a loser and we must pay \$1. (The notation $a \setminus b$, read "a divides b," means that b is an exact multiple of a; Chapter 4 investigates this relation carefully.) Can we expect to make money if we play this game?

We can compute the average winnings — that is, the amount we'll win (or lose) per play — by first counting the number W of winners and the number $L = 1000 - W$ of losers. If each number comes up once during 1000 plays, we win $5W$ dollars and lose L dollars, so the average winnings will be

$$\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6W - 1000}{1000}.$$

If there are 167 or more winners, we have the advantage; otherwise the advantage is with the house.

How can we count the number of winners among 1 through 1000? It's not hard to spot a pattern. The numbers from 1 through $2^3 - 1 = 7$ are all winners because $\lfloor \sqrt[3]{n} \rfloor = 1$ for each. Among the numbers $2^3 = 8$ through $3^3 - 1 = 26$, only the even numbers are winners. And among $3^3 = 27$ through $4^3 - 1 = 63$, only those divisible by 3 are. And so on.

Just like we can remember the date of Columbus's departure by singing, "In fourteen hundred and ninety-three/ Columbus sailed the deep blue sea."

(A poll of the class at this point showed that 28 students thought it was a bad idea to play, 13 wanted to gamble, and the rest were too confused to answer.)

(So we hit them with the Concrete Math Club.)

The whole setup can be analyzed systematically if we use the summation techniques of Chapter 2, taking advantage of Iverson's convention about logical statements evaluating to 0 or 1:

$$\begin{aligned}
 W &= \sum_{n=1}^{1000} [n \text{ is a winner}] \\
 &= \sum_{1 \leq n \leq 1000} [\lfloor \sqrt[3]{n} \rfloor \setminus n] = \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor] [k \setminus n] [1 \leq n \leq 1000] \\
 &= \sum_{k,m,n} [k^3 \leq n < (k+1)^3] [n = km] [1 \leq n \leq 1000] \\
 &= 1 + \sum_{k,m} [k^3 \leq km < (k+1)^3] [1 \leq k < 10] \\
 &= 1 + \sum_{k,m} [m \in [k^2 \dots (k+1)^3/k]] [1 \leq k < 10] \\
 &= 1 + \sum_{1 \leq k < 10} (\lceil k^2 + 3k + 3 + 1/k \rceil - \lceil k^2 \rceil) \\
 &= 1 + \sum_{1 \leq k < 10} (3k + 4) = 1 + \frac{7+31}{2} \cdot 9 = 172.
 \end{aligned}$$

This derivation merits careful study. Notice that line 6 uses our formula (3.12) for the number of integers in a half-open interval. The only "difficult" maneuver is the decision made between lines 3 and 4 to treat $n = 1000$ as a special case. (The inequality $k^3 \leq n < (k+1)^3$ does not combine easily with $1 \leq n \leq 1000$ when $k = 10$.) In general, boundary conditions tend to be the most critical part of \sum -manipulations.

True.

Where did you say this casino is?

The bottom line says that $W = 172$; hence our formula for average winnings per play reduces to $(6 \cdot 172 - 1000)/1000$ dollars, which is 3.2 cents. We can expect to be about \$3.20 richer after making 100 bets of \$1 each. (Of course, the house may have made some numbers more equal than others.)

The casino problem we just solved is a dressed-up version of the more mundane question, "How many integers n , where $1 \leq n \leq 1000$, satisfy the relation $\lfloor \sqrt[3]{n} \rfloor \setminus n$?" Mathematically the two questions are the same. But sometimes it's a good idea to dress up a problem. We get to use more vocabulary (like "winners" and "losers"), which helps us to understand what's going on.

Let's get general. Suppose we change 1000 to 1000000, or to an even larger number, N . (We assume that the casino has connections and can get a bigger wheel.) Now how many winners are there?

The same argument applies, but we need to deal more carefully with the largest value of k , which we can call K for convenience:

$$K = \lfloor \sqrt[3]{N} \rfloor.$$

(Previously K was 10.) The total number of winners for general N comes to

$$\begin{aligned} W &= \sum_{1 \leq k < K} (3k + 4) + \sum_m [K^3 \leq Km \leq N] \\ &= \frac{1}{2}(7 + 3K + 1)(K - 1) + \sum_m [m \in [K^2 \dots N/K]] \\ &= \frac{3}{2}K^2 + \frac{5}{2}K - 4 + \sum_m [m \in [K^2 \dots N/K]]. \end{aligned}$$

We know that the remaining sum is $\lfloor N/K \rfloor - \lceil K^2 \rceil + 1 = \lfloor N/K \rfloor - K^2 + 1$; hence the formula

$$W = \lfloor N/K \rfloor + \frac{1}{2}K^2 + \frac{5}{2}K - 3, \quad K = \lfloor \sqrt[3]{N} \rfloor \quad (3.13)$$

gives the general answer for a wheel of size N .

The first two terms of this formula are approximately $N^{2/3} + \frac{1}{2}N^{2/3} = \frac{3}{2}N^{2/3}$, and the other terms are much smaller in comparison, when N is large. In Chapter 9 we'll learn how to derive expressions like

$$W = \frac{3}{2}N^{2/3} + O(N^{1/3}),$$

where $O(N^{1/3})$ stands for a quantity that is no more than a constant times $N^{1/3}$. Whatever the constant is, we know that it's independent of N ; so for large N the contribution of the O -term to W will be quite small compared with $\frac{3}{2}N^{2/3}$. For example, the following table shows how close $\frac{3}{2}N^{2/3}$ is to W :

N	$\frac{3}{2}N^{2/3}$	W	% error
1,000	150.0	172	12.791
10,000	696.2	746	6.670
100,000	3231.7	3343	3.331
1,000,000	15000.0	15247	1.620
10,000,000	69623.8	70158	0.761
100,000,000	323165.2	324322	0.357
1,000,000,000	1500000.0	1502497	0.166

It's a pretty good approximation.

Approximate formulas are useful because they're simpler than formulas with floors and ceilings. However, the exact truth is often important, too, especially for the smaller values of N that tend to occur in practice. For example, the casino owner may have falsely assumed that there are only $\frac{3}{2}N^{2/3} = 150$ winners when $N = 1000$ (in which case there would be a 10¢ advantage for the house).

Our last application in this section looks at so-called spectra. We define the *spectrum* of a positive real number α to be an infinite multiset of integers,

$$\text{Spec}(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \dots\}.$$

(A multiset is like a set but it can have repeated elements.) For example, the spectrum of $1/2$ starts out $\{0, 1, 1, 2, 2, 3, 3, \dots\}$.

It's easy to prove that no two spectra are equal—that $\alpha \neq \beta$ implies $\text{Spec}(\alpha) \neq \text{Spec}(\beta)$. For, assuming without loss of generality that $\alpha < \beta$, there's a positive integer m such that $m(\beta - \alpha) \geq 1$. (In fact, any $m \geq \lceil 1/(\beta - \alpha) \rceil$ will do; but we needn't show off our knowledge of floors and ceilings all the time.) Hence $m\beta - m\alpha \geq 1$, and $\lfloor m\beta \rfloor > \lfloor m\alpha \rfloor$. Thus $\text{Spec}(\beta)$ has fewer than m elements $\leq \lfloor m\alpha \rfloor$, while $\text{Spec}(\alpha)$ has at least m .

Spectra have many beautiful properties. For example, consider the two multisets

$$\begin{aligned} \text{Spec}(\sqrt{2}) &= \{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, 22, 24, \dots\}, \\ \text{Spec}(2 + \sqrt{2}) &= \{3, 6, 10, 13, 17, 20, 23, 27, 30, 34, 37, 40, 44, 47, 51, \dots\}. \end{aligned}$$

It's easy to calculate $\text{Spec}(\sqrt{2})$ with a pocket calculator, and the n th element of $\text{Spec}(2 + \sqrt{2})$ is just $2n$ more than the n th element of $\text{Spec}(\sqrt{2})$, by (3.6). A closer look shows that these two spectra are also related in a much more surprising way: It seems that any number missing from one is in the other, but that no number is in both! And it's true: The positive integers are the disjoint union of $\text{Spec}(\sqrt{2})$ and $\text{Spec}(2 + \sqrt{2})$. We say that these spectra form a *partition* of the positive integers.

To prove this assertion, we will count how many of the elements of $\text{Spec}(\sqrt{2})$ are $\leq n$, and how many of the elements of $\text{Spec}(2 + \sqrt{2})$ are $\leq n$. If the total is n , for each n , these two spectra do indeed form a partition.

Whenever $\alpha > 0$, the number of elements in $\text{Spec}(\alpha)$ that are $\leq n$ is

$$\begin{aligned} N(\alpha, n) &= \sum_{k>0} [\lfloor k\alpha \rfloor \leq n] \\ &= \sum_{k>0} [\lfloor k\alpha \rfloor < n + 1] \\ &= \sum_{k>0} [k\alpha < n + 1] \\ &= \sum_k [0 < k < (n + 1)/\alpha] \\ &= \lceil (n + 1)/\alpha \rceil - 1. \end{aligned} \tag{3.14}$$

... without lots of generality...

"If x be an incommensurable number less than unity, one of the series of quantities m/x , $m/(1-x)$, where m is a whole number, can be found which shall lie between any given consecutive integers, and but one such quantity can be found."
—Rayleigh [304]

Right, because exactly one of the counts must increase when n increases by 1.

This derivation has two special points of interest. First, it uses the law

$$m \leq n \iff m < n + 1, \quad \text{integers } m \text{ and } n \quad (3.15)$$

to change ' \leq ' to '<', so that the floor brackets can be removed by (3.7). Also — and this is more subtle — it sums over the range $k > 0$ instead of $k \geq 1$, because $(n + 1)/\alpha$ might be less than 1 for certain n and α . If we had tried to apply (3.12) to determine the number of integers in $[1..(n + 1)/\alpha)$, rather than the number of integers in $(0..(n + 1)/\alpha)$, we would have gotten the right answer; but our derivation would have been faulty because the conditions of applicability wouldn't have been met.

Good, we have a formula for $N(\alpha, n)$. Now we can test whether or not $\text{Spec}(\sqrt{2})$ and $\text{Spec}(2 + \sqrt{2})$ partition the positive integers, by testing whether or not $N(\sqrt{2}, n) + N(2 + \sqrt{2}, n) = n$ for all integers $n > 0$, using (3.14):

$$\begin{aligned} \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor - 1 + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor - 1 &= n \\ \iff \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor &= n, && \text{by (3.2);} \\ \iff \frac{n+1}{\sqrt{2}} - \left\{ \frac{n+1}{\sqrt{2}} \right\} + \frac{n+1}{2+\sqrt{2}} - \left\{ \frac{n+1}{2+\sqrt{2}} \right\} &= n, && \text{by (3.8).} \end{aligned}$$

Everything simplifies now because of the neat identity

$$\frac{1}{\sqrt{2}} + \frac{1}{2 + \sqrt{2}} = 1;$$

our condition reduces to testing whether or not

$$\left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\} = 1,$$

for all $n > 0$. And we win, because these are the fractional parts of two noninteger numbers that add up to the integer $n + 1$. A partition it is.

3.3 FLOOR/CEILING RECURRENCES

Floors and ceilings add an interesting new dimension to the study of recurrence relations. Let's look first at the recurrence

$$\begin{aligned} K_0 &= 1; \\ K_{n+1} &= 1 + \min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}), \quad \text{for } n \geq 0. \end{aligned} \quad (3.16)$$

Thus, for example, K_1 is $1 + \min(2K_0, 3K_0) = 3$; the sequence begins 1, 3, 3, 4, 7, 7, 7, 9, 9, 10, 13, ... One of the authors of this book has modestly decided to call these the Knuth numbers.

Exercise 25 asks for a proof or disproof that $K_n \geq n$, for all $n \geq 0$. The first few K 's just listed do satisfy the inequality, so there's a good chance that it's true in general. Let's try an induction proof: The basis $n = 0$ comes directly from the defining recurrence. For the induction step, we assume that the inequality holds for all values up through some fixed nonnegative n , and we try to show that $K_{n+1} \geq n + 1$. From the recurrence we know that $K_{n+1} = 1 + \min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor})$. The induction hypothesis tells us that $2K_{\lfloor n/2 \rfloor} \geq 2\lfloor n/2 \rfloor$ and $3K_{\lfloor n/3 \rfloor} \geq 3\lfloor n/3 \rfloor$. However, $2\lfloor n/2 \rfloor$ can be as small as $n - 1$, and $3\lfloor n/3 \rfloor$ can be as small as $n - 2$. The most we can conclude from our induction hypothesis is that $K_{n+1} \geq 1 + (n - 2)$; this falls far short of $K_{n+1} \geq n + 1$.

We now have reason to worry about the truth of $K_n \geq n$, so let's try to disprove it. If we can find an n such that either $2K_{\lfloor n/2 \rfloor} < n$ or $3K_{\lfloor n/3 \rfloor} < n$, or in other words such that

$$K_{\lfloor n/2 \rfloor} < n/2 \quad \text{or} \quad K_{\lfloor n/3 \rfloor} < n/3,$$

we will have $K_{n+1} < n + 1$. Can this be possible? We'd better not give the answer away here, because that will spoil exercise 25.

Recurrence relations involving floors and/or ceilings arise often in computer science, because algorithms based on the important technique of "divide and conquer" often reduce a problem of size n to the solution of similar problems of integer sizes that are fractions of n . For example, one way to sort n records, if $n > 1$, is to divide them into two approximately equal parts, one of size $\lceil n/2 \rceil$ and the other of size $\lfloor n/2 \rfloor$. (Notice, incidentally, that

$$n = \lceil n/2 \rceil + \lfloor n/2 \rfloor; \tag{3.17}$$

this formula comes in handy rather often.) After each part has been sorted separately (by the same method, applied recursively), we can merge the records into their final order by doing at most $n - 1$ further comparisons. Therefore the total number of comparisons performed is at most $f(n)$, where

$$\begin{aligned} f(1) &= 0; \\ f(n) &= f(\lceil n/2 \rceil) + f(\lfloor n/2 \rfloor) + n - 1, \quad \text{for } n > 1. \end{aligned} \tag{3.18}$$

A solution to this recurrence appears in exercise 34.

The Josephus problem of Chapter 1 has a similar recurrence, which can be cast in the form

$$\begin{aligned} J(1) &= 1; \\ J(n) &= 2J(\lfloor n/2 \rfloor) - (-1)^n, \quad \text{for } n > 1. \end{aligned}$$

We've got more tools to work with than we had in Chapter 1, so let's consider the more authentic Josephus problem in which every third person is eliminated, instead of every second. If we apply the methods that worked in Chapter 1 to this more difficult problem, we wind up with a recurrence like

$$J_3(n) = \left(\left\lceil \frac{3}{2} J_3 \left(\lfloor \frac{2}{3} n \rfloor \right) + a_n \right\rceil \bmod n \right) + 1,$$

where 'mod' is a function that we will be studying shortly, and where we have $a_n = -2, +1$, or $-\frac{1}{2}$ according as $n \bmod 3 = 0, 1$, or 2 . But this recurrence is too horrible to pursue.

There's another approach to the Josephus problem that gives a much better setup. Whenever a person is passed over, we can assign a new number. Thus, 1 and 2 become $n + 1$ and $n + 2$, then 3 is executed; 4 and 5 become $n + 3$ and $n + 4$, then 6 is executed; ...; $3k + 1$ and $3k + 2$ become $n + 2k + 1$ and $n + 2k + 2$, then $3k + 3$ is executed; ... then $3n$ is executed (or left to survive). For example, when $n = 10$ the numbers are

1	2	3	4	5	6	7	8	9	10
11	12		13	14		15	16		17
18			19	20			21		22
			23	24					25
			26						27
			28						
			29						
			30						

The k th person eliminated ends up with number $3k$. So we can figure out who the survivor is if we can figure out the original number of person number $3n$.

If $N > n$, person number N must have had a previous number, and we can find it as follows: We have $N = n + 2k + 1$ or $N = n + 2k + 2$, hence $k = \lfloor (N - n - 1)/2 \rfloor$; the previous number was $3k + 1$ or $3k + 2$, respectively. That is, it was $3k + (N - n - 2k) = k + N - n$. Hence we can calculate the survivor's number $J_3(n)$ as follows:

```

N := 3n;
while N > n do N := ⌊  $\frac{N - n - 1}{2}$  ⌋ + N - n;
J3(n) := N.

```

This is not a closed form for $J_3(n)$; it's not even a recurrence. But at least it tells us how to calculate the answer reasonably fast, if n is large.

*“Not too slow,
not too fast.”*
—L. Armstrong

Fortunately there's a way to simplify this algorithm if we use the variable $D = 3n + 1 - N$ in place of N . (This change in notation corresponds to assigning numbers from $3n$ down to 1 , instead of from 1 up to $3n$; it's sort of like a countdown.) Then the complicated assignment to N becomes

$$\begin{aligned} D &:= 3n + 1 - \left(\left\lfloor \frac{(3n + 1 - D) - n - 1}{2} \right\rfloor + (3n + 1 - D) - n \right) \\ &= n + D - \left\lfloor \frac{2n - D}{2} \right\rfloor = D - \left\lfloor \frac{-D}{2} \right\rfloor = D + \left\lceil \frac{D}{2} \right\rceil = \left\lceil \frac{3}{2}D \right\rceil, \end{aligned}$$

and we can rewrite the algorithm as follows:

```
D := 1;
while D ≤ 2n do D := ⌈ $\frac{3}{2}$ D⌉;
J3(n) := 3n + 1 - D.
```

Aha! This looks much nicer, because n enters the calculation in a very simple way. In fact, we can show by the same reasoning that the survivor $J_q(n)$ when every q th person is eliminated can be calculated as follows:

```
D := 1;
while D ≤ (q - 1)n do D := ⌈ $\frac{q}{q-1}$ D⌉;
Jq(n) := qn + 1 - D. (3.19)
```

In the case $q = 2$ that we know so well, this makes D grow to 2^{m+1} when $n = 2^m + 1$; hence $J_2(n) = 2(2^m + 1) + 1 - 2^{m+1} = 2l + 1$. Good.

The recipe in (3.19) computes a sequence of integers that can be defined by the following recurrence:

$$\begin{aligned} D_0^{(q)} &= 1; \\ D_n^{(q)} &= \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil \quad \text{for } n > 0. \end{aligned} \tag{3.20}$$

These numbers don't seem to relate to any familiar functions in a simple way, except when $q = 2$; hence they probably don't have a nice closed form. But if we're willing to accept the sequence $D_n^{(q)}$ as "known," then it's easy to describe the solution to the generalized Josephus problem: The survivor $J_q(n)$ is $qn + 1 - D_k^{(q)}$, where k is as small as possible such that $D_k^{(q)} > (q - 1)n$.

"Known" like, say, harmonic numbers. A. M. Odlyzko and H. S. Wilf have shown [283] that

$$D_n^{(3)} = \left\lfloor \left(\frac{3}{2}\right)^n C \right\rfloor,$$

where

$$C \approx 1.622270503.$$

3.4 'MOD': THE BINARY OPERATION

The quotient of n divided by m is $\lfloor n/m \rfloor$, when m and n are positive integers. It's handy to have a simple notation also for the remainder of this

division, and we call it ‘ $n \bmod m$ ’. The basic formula

$$n = m \underbrace{\lfloor n/m \rfloor}_{\text{quotient}} + \underbrace{n \bmod m}_{\text{remainder}}$$

tells us that we can express $n \bmod m$ as $n - m\lfloor n/m \rfloor$. We can generalize this to negative integers, and in fact to arbitrary real numbers:

$$x \bmod y = x - y\lfloor x/y \rfloor, \quad \text{for } y \neq 0. \quad (3.21)$$

This defines ‘mod’ as a binary operation, just as addition and subtraction are binary operations. Mathematicians have used mod this way informally for a long time, taking various quantities mod 10, mod 2π , and so on, but only in the last twenty years has it caught on formally. Old notion, new notation.

We can easily grasp the intuitive meaning of $x \bmod y$, when x and y are positive real numbers, if we imagine a circle of circumference y whose points have been assigned real numbers in the interval $[0..y)$. If we travel a distance x around the circle, starting at 0, we end up at $x \bmod y$. (And the number of times we encounter 0 as we go is $\lfloor x/y \rfloor$.)

When x or y is negative, we need to look at the definition carefully in order to see exactly what it means. Here are some integer-valued examples:

$$\begin{aligned} 5 \bmod 3 &= 5 - 3\lfloor 5/3 \rfloor &&= 2; \\ 5 \bmod -3 &= 5 - (-3)\lfloor 5/(-3) \rfloor &&= -1; \\ -5 \bmod 3 &= -5 - 3\lfloor -5/3 \rfloor &&= 1; \\ -5 \bmod -3 &= -5 - (-3)\lfloor -5/(-3) \rfloor &&= -2. \end{aligned}$$

The number after ‘mod’ is called the *modulus*; nobody has yet decided what to call the number before ‘mod’. In applications, the modulus is usually positive, but the definition makes perfect sense when the modulus is negative. In both cases the value of $x \bmod y$ is between 0 and the modulus:

$$\begin{aligned} 0 &\leq x \bmod y < y, && \text{for } y > 0; \\ 0 &\geq x \bmod y > y, && \text{for } y < 0. \end{aligned}$$

What about $y = 0$? Definition (3.21) leaves this case undefined, in order to avoid division by zero, but to be complete we can define

$$x \bmod 0 = x. \quad (3.22)$$

This convention preserves the property that $x \bmod y$ always differs from x by a multiple of y . (It might seem more natural to make the function continuous at 0, by defining $x \bmod 0 = \lim_{y \rightarrow 0} x \bmod y = 0$. But we’ll see in Chapter 4

Why do they call it ‘mod’: The Binary Operation? Stay tuned to find out in the next, exciting, chapter!

Beware of computer languages that use another definition.

How about calling the other number the modumor?

that this would be much less useful. Continuity is not an important aspect of the mod operation.)

We've already seen one special case of mod in disguise, when we wrote x in terms of its integer and fractional parts, $x = \lfloor x \rfloor + \{x\}$. The fractional part can also be written $x \bmod 1$, because we have

$$x = \lfloor x \rfloor + x \bmod 1.$$

Notice that parentheses aren't needed in this formula; we take mod to bind more tightly than addition or subtraction.

The floor function has been used to define mod, and the ceiling function hasn't gotten equal time. We could perhaps use the ceiling to define a mod analog like

$$x \text{ mumble } y = y \lceil x/y \rceil - x;$$

in our circle analogy this represents the distance the traveler needs to continue, after going a distance x , to get back to the starting point 0. But of course we'd need a better name than 'mumble'. If sufficient applications come along, an appropriate name will probably suggest itself.

The distributive law is mod's most important algebraic property: We have

$$c(x \bmod y) = (cx) \bmod (cy) \tag{3.23}$$

for all real c , x , and y . (Those who like mod to bind less tightly than multiplication may remove the parentheses from the right side here, too.) It's easy to prove this law from definition (3.21), since

$$c(x \bmod y) = c(x - y \lfloor x/y \rfloor) = cx - cy \lfloor cx/cy \rfloor = cx \bmod cy,$$

if $cy \neq 0$; and the zero-modulus cases are trivially true. Our four examples using ± 5 and ± 3 illustrate this law twice, with $c = -1$. An identity like (3.23) is reassuring, because it gives us reason to believe that 'mod' has not been defined improperly.

In the remainder of this section, we'll consider an application in which 'mod' turns out to be helpful although it doesn't play a central role. The problem arises frequently in a variety of situations: We want to partition n things into m groups as equally as possible.

Suppose, for example, that we have n short lines of text that we'd like to arrange in m columns. For æsthetic reasons, we want the columns to be arranged in decreasing order of length (actually nonincreasing order); and the lengths should be approximately the same — no two columns should differ by

There was a time in the 70s when 'mod' was the fashion.

Maybe the new mumble function should be called 'punk'?

No—I like 'mumble'.

Notice that

$x \text{ mumble } y = (-x) \bmod y$.

The remainder, eh?

more than one line's worth of text. If 37 lines of text are being divided into five columns, we would therefore prefer the arrangement on the right:

8					5					8					8					7					7																																																																																									
line 1	line 9	line 17	line 25	line 33	line 1	line 9	line 17	line 24	line 31	line 1	line 9	line 17	line 24	line 31	line 2	line 10	line 18	line 25	line 32	line 2	line 10	line 18	line 25	line 32	line 3	line 11	line 19	line 27	line 35	line 3	line 11	line 19	line 26	line 33	line 3	line 11	line 19	line 26	line 33	line 4	line 12	line 20	line 28	line 36	line 4	line 12	line 20	line 27	line 34	line 4	line 12	line 20	line 27	line 34	line 5	line 13	line 21	line 29	line 37	line 5	line 13	line 21	line 28	line 35	line 5	line 13	line 21	line 28	line 35	line 6	line 14	line 22	line 30		line 6	line 14	line 22	line 29	line 36	line 6	line 14	line 22	line 29	line 36	line 7	line 15	line 23	line 31		line 7	line 15	line 23	line 30	line 37	line 7	line 15	line 23	line 30	line 37	line 8	line 16	line 24	line 32		line 8	line 16				line 8	line 16			

Furthermore we want to distribute the lines of text columnwise — first deciding how many lines go into the first column and then moving on to the second, the third, and so on — because that's the way people read. Distributing row by row would give us the correct number of lines in each column, but the ordering would be wrong. (We would get something like the arrangement on the right, but column 1 would contain lines 1, 6, 11, . . . , 36, instead of lines 1, 2, 3, . . . , 8 as desired.)

A row-by-row distribution strategy can't be used, but it does tell us how many lines to put in each column. If n is not a multiple of m , the row-by-row procedure makes it clear that the long columns should each contain $\lceil n/m \rceil$ lines, and the short columns should each contain $\lfloor n/m \rfloor$. There will be exactly $n \bmod m$ long columns (and, as it turns out, there will be exactly $n \bmod m$ mumble m short ones).

Let's generalize the terminology and talk about 'things' and 'groups' instead of 'lines' and 'columns'. We have just decided that the first group should contain $\lceil n/m \rceil$ things; therefore the following sequential distribution scheme ought to work: To distribute n things into m groups, when $m > 0$, put $\lceil n/m \rceil$ things into one group, then use the same procedure recursively to put the remaining $n' = n - \lceil n/m \rceil$ things into $m' = m - 1$ additional groups.

For example, if $n = 314$ and $m = 6$, the distribution goes like this:

remaining things	remaining groups	$\lceil \text{things/groups} \rceil$
314	6	53
261	5	53
208	4	52
156	3	52
104	2	52
52	1	52

It works. We get groups of approximately the same size, even though the divisor keeps changing.

Why does it work? In general we can suppose that $n = qm + r$, where $q = \lfloor n/m \rfloor$ and $r = n \bmod m$. The process is simple if $r = 0$: We put $\lfloor n/m \rfloor = q$ things into the first group and replace n by $n' = n - q$, leaving

$n' = qm'$ things to put into the remaining $m' = m - 1$ groups. And if $r > 0$, we put $\lceil n/m \rceil = q + 1$ things into the first group and replace n by $n' = n - q - 1$, leaving $n' = qm' + r - 1$ things for subsequent groups. The new remainder is $r' = r - 1$, but q stays the same. It follows that there will be r groups with $q + 1$ things, followed by $m - r$ groups with q things.

How many things are in the k th group? We'd like a formula that gives $\lceil n/m \rceil$ when $k \leq n \bmod m$, and $\lfloor n/m \rfloor$ otherwise. It's not hard to verify that

$$\left\lceil \frac{n - k + 1}{m} \right\rceil$$

has the desired properties, because this reduces to $q + \lceil (r - k + 1)/m \rceil$ if we write $n = qm + r$ as in the preceding paragraph; here $q = \lfloor n/m \rfloor$. We have $\lceil (r - k + 1)/m \rceil = \lfloor k/m \rfloor$, if $1 \leq k \leq m$ and $0 \leq r < m$. Therefore we can write an identity that expresses the partition of n into m as-equal-as-possible parts in nonincreasing order:

$$n = \left\lceil \frac{n}{m} \right\rceil + \left\lceil \frac{n-1}{m} \right\rceil + \cdots + \left\lceil \frac{n-m+1}{m} \right\rceil. \quad (3.24)$$

This identity is valid for all positive integers m , and for all integers n (whether positive, negative, or zero). We have already encountered the case $m = 2$ in (3.17), although we wrote it in a slightly different form, $n = \lceil n/2 \rceil + \lfloor n/2 \rfloor$.

If we had wanted the parts to be in nondecreasing order, with the small groups coming before the larger ones, we could have proceeded in the same way but with $\lfloor n/m \rfloor$ things in the first group. Then we would have derived the corresponding identity

$$n = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + \cdots + \left\lfloor \frac{n+m-1}{m} \right\rfloor. \quad (3.25)$$

It's possible to convert between (3.25) and (3.24) by using either (3.4) or the identity of exercise 12.

Now if we replace n in (3.25) by $\lfloor mx \rfloor$, and apply rule (3.11) to remove floors inside of floors, we get an identity that holds for all real x :

$$\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + \cdots + \left\lfloor x + \frac{m-1}{m} \right\rfloor. \quad (3.26)$$

This is rather amazing, because the floor function is an integer approximation of a real value, but the single approximation on the left equals the sum of a bunch of them on the right. If we assume that $\lfloor x \rfloor$ is roughly $x - \frac{1}{2}$ on the average, the left-hand side is roughly $mx - \frac{1}{2}$, while the right-hand side comes to roughly $(x - \frac{1}{2}) + (x - \frac{1}{2} + \frac{1}{m}) + \cdots + (x - \frac{1}{2} + \frac{m-1}{m}) = mx - \frac{1}{2}$; the sum of all these rough approximations turns out to be exact!

Some claim that it's too dangerous to replace anything by an mx .

3.5 FLOOR/CEILING SUMS

Equation (3.26) demonstrates that it's possible to get a closed form for at least one kind of sum that involves $\lfloor \cdot \rfloor$. Are there others? Yes. The trick that usually works in such cases is to get rid of the floor or ceiling by introducing a new variable.

For example, let's see if it's possible to do the sum

$$\sum_{0 \leq k < n} \lfloor \sqrt{k} \rfloor$$

in closed form. One idea is to introduce the variable $m = \lfloor \sqrt{k} \rfloor$; we can do this "mechanically" by proceeding as we did in the roulette problem:

$$\begin{aligned} \sum_{0 \leq k < n} \lfloor \sqrt{k} \rfloor &= \sum_{k, m \geq 0} m[k < n][m = \lfloor \sqrt{k} \rfloor] \\ &= \sum_{k, m \geq 0} m[k < n][m \leq \sqrt{k} < m + 1] \\ &= \sum_{k, m \geq 0} m[k < n][m^2 \leq k < (m + 1)^2] \\ &= \sum_{k, m \geq 0} m[m^2 \leq k < (m + 1)^2 \leq n] \\ &\quad + \sum_{k, m \geq 0} m[m^2 \leq k < n < (m + 1)^2]. \end{aligned}$$

Once again the boundary conditions are a bit delicate. Let's assume first that $n = a^2$ is a perfect square. Then the second sum is zero, and the first can be evaluated by our usual routine:

$$\begin{aligned} &\sum_{k, m \geq 0} m[m^2 \leq k < (m + 1)^2 \leq a^2] \\ &= \sum_{m \geq 0} m((m + 1)^2 - m^2)[m + 1 \leq a] \\ &= \sum_{m \geq 0} m(2m + 1)[m < a] \\ &= \sum_{m \geq 0} (2m^2 + 3m^1)[m < a] \\ &= \sum_0^a (2m^2 + 3m^1) \delta m \\ &= \frac{2}{3}a(a - 1)(a - 2) + \frac{3}{2}a(a - 1) = \frac{1}{6}(4a + 1)a(a - 1). \end{aligned}$$

*Falling powers
make the sum come
tumbling down.*

In the general case we can let $a = \lfloor \sqrt{n} \rfloor$; then we merely need to add the terms for $a^2 \leq k < n$, which are all equal to a , so they sum to $(n - a^2)a$. This gives the desired closed form,

$$\sum_{0 \leq k < n} \lfloor \sqrt{k} \rfloor = na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a, \quad a = \lfloor \sqrt{n} \rfloor. \quad (3.27)$$

Another approach to such sums is to replace an expression of the form $\lfloor x \rfloor$ by $\sum_j [1 \leq j \leq x]$; this is legal whenever $x \geq 0$. Here's how that method works in the sum of $\lfloor \text{square roots} \rfloor$, if we assume for convenience that $n = a^2$:

$$\begin{aligned} \sum_{0 \leq k < a^2} \lfloor \sqrt{k} \rfloor &= \sum_{j,k} [1 \leq j \leq \sqrt{k}] [0 \leq k < a^2] \\ &= \sum_{1 \leq j < a} \sum_k [j^2 \leq k < a^2] \\ &= \sum_{1 \leq j < a} (a^2 - j^2) = a^3 - \frac{1}{3}a(a + \frac{1}{2})(a + 1), \quad \text{integer } a. \end{aligned}$$

Now here's another example where a change of variable leads to a transformed sum. A remarkable theorem was discovered independently by three mathematicians—Bohl [34], Sierpiński [326], and Weyl [368]—at about the same time in 1909: If α is irrational then the fractional parts $\{n\alpha\}$ are very uniformly distributed between 0 and 1, as $n \rightarrow \infty$. One way to state this is that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq k < n} f(\{k\alpha\}) = \int_0^1 f(x) dx \quad (3.28)$$

for all irrational α and all bounded functions f that are continuous almost everywhere. For example, the *average* value of $\{n\alpha\}$ can be found by setting $f(x) = x$; we get $\frac{1}{2}$. (That's exactly what we might expect; but it's nice to know that it is really, provably true, no matter how irrational α is.)

The theorem of Bohl, Sierpiński, and Weyl is proved by approximating $f(x)$ above and below by “step functions,” which are linear combinations of the simple functions

$$f_v(x) = [0 \leq x < v]$$

when $0 \leq v \leq 1$. Our purpose here is not to prove the theorem; that's a job for calculus books. But let's try to figure out the basic reason why it holds, by seeing how well it works in the special case $f(x) = f_v(x)$. In other words, let's try to see how close the sum

$$\sum_{0 \leq k < n} [\{k\alpha\} < v]$$

gets to the “ideal” value nv , when n is large and α is irrational.

Warning: This stuff is fairly advanced. Better skim the next two pages on first reading; they aren't crucial.

—Friendly TA

Start
Skimming

For this purpose we define the *discrepancy* $D(\alpha, n)$ to be the maximum absolute value, over all $0 \leq v \leq 1$, of the sum

$$s(\alpha, n, v) = \sum_{0 \leq k < n} \left(\lceil [k\alpha] \rceil - v \right). \tag{3.29}$$

Our goal is to show that $D(\alpha, n)$ is “not too large” when compared with n , by showing that $|s(\alpha, n, v)|$ is always reasonably small when α is irrational. We can assume without loss of generality that $0 < \alpha < 1$.

First we can rewrite $s(\alpha, n, v)$ in simpler form, then introduce a new index variable j :

$$\begin{aligned} \sum_{0 \leq k < n} \left(\lceil [k\alpha] \rceil - v \right) &= \sum_{0 \leq k < n} ([k\alpha] - \lfloor k\alpha - v \rfloor - v) \\ &= -nv + \sum_{0 \leq k < n} \sum_j [k\alpha - v < j \leq k\alpha] \\ &= -nv + \sum_{0 \leq j < \lceil n\alpha \rceil} \sum_{k < n} [j\alpha^{-1} \leq k < (j+v)\alpha^{-1}]. \end{aligned}$$

If we’re lucky, we can do the sum on k . But we ought to introduce some new variables, so that the formula won’t be such a mess. Let us write

$$\begin{aligned} a &= \lfloor \alpha^{-1} \rfloor, & \alpha^{-1} &= a + \alpha'; \\ b &= \lceil v\alpha^{-1} \rceil, & v\alpha^{-1} &= b - v'. \end{aligned}$$

Right, name and conquer. The change of variable from k to j is the main point. —Friendly TA

Thus $\alpha' = \{\alpha^{-1}\}$ is the fractional part of α^{-1} , and v' is the mumble-fractional part of $v\alpha^{-1}$.

Once again the boundary conditions are our only source of grief. For now, let’s forget the restriction ‘ $k < n$ ’ and evaluate the sum on k without it:

$$\begin{aligned} \sum_k [k \in [j\alpha^{-1} .. (j+v)\alpha^{-1}]] &= \lceil (j+v)(a + \alpha') \rceil - \lceil j(a + \alpha') \rceil \\ &= b + \lceil j\alpha' - v' \rceil - \lceil j\alpha' \rceil. \end{aligned}$$

OK, that’s pretty simple; we plug it in and plug away:

$$s(\alpha, n, v) = -nv + \lceil n\alpha \rceil b + \sum_{0 \leq j < \lceil n\alpha \rceil} (\lceil j\alpha' - v' \rceil - \lceil j\alpha' \rceil) - S, \tag{3.30}$$

where S is a correction for the cases with $k \geq n$ that we have failed to exclude. The quantity $j\alpha'$ will be an integer only when $j = 0$, since α (hence α') is irrational; and $j\alpha' - v'$ will be an integer for at most one value of j . So we

can change the ceiling terms to floors:

$$s(\alpha, n, v) = -nv + \lceil n\alpha \rceil b - \sum_{0 \leq j < \lceil n\alpha \rceil} (\lfloor j\alpha' \rfloor - \lfloor j\alpha' - v' \rfloor) - S + \{0 \text{ or } 1\}.$$

(The formula {0 or 1} stands for something that's either 0 or 1; we needn't commit ourselves, because the details don't really matter.)

Interesting. Instead of a closed form, we're getting a sum that looks rather like $s(\alpha, n, v)$ but with different parameters: α' instead of α , $\lceil n\alpha \rceil$ instead of n , and v' instead of v . So we'll have a recurrence for $s(\alpha, n, v)$, which (hopefully) will lead to a recurrence for the discrepancy $D(\alpha, n)$. This means we want to get

$$s(\alpha', \lceil n\alpha \rceil, v') = \sum_{0 \leq j < \lceil n\alpha \rceil} (\lfloor j\alpha' \rfloor - \lfloor j\alpha' - v' \rfloor - v')$$

into the act:

$$s(\alpha, n, v) = -nv + \lceil n\alpha \rceil b - \lceil n\alpha \rceil v' - s(\alpha', \lceil n\alpha \rceil, v') - S + \{0 \text{ or } 1\}.$$

Recalling that $b - v' = v\alpha^{-1}$, we see that everything will simplify beautifully if we replace $\lceil n\alpha \rceil(b - v')$ by $n\alpha(b - v') = nv$:

$$s(\alpha, n, v) = -s(\alpha', \lceil n\alpha \rceil, v') - S + \epsilon + \{0 \text{ or } 1\}.$$

Here ϵ is a positive error less than $v\alpha^{-1}$. Exercise 18 proves that S is, similarly, between 0 and $\lceil v\alpha^{-1} \rceil$. And we can remove the term for $j = \lceil n\alpha \rceil - 1 = \lfloor n\alpha \rfloor$ from the sum, since it contributes either v' or $v' - 1$. Hence, if we take the maximum of absolute values over all v , we get

$$D(\alpha, n) \leq D(\alpha', \lfloor n\alpha \rfloor) + \alpha^{-1} + 2. \quad (3.31)$$

The methods we'll learn in succeeding chapters will allow us to conclude from this recurrence that $D(\alpha, n)$ is always much smaller than n , when n is sufficiently large. Hence theorem (3.28) is true; however, convergence to the limit is not always very fast. (See exercises 9.45 and 9.61.)

Whew; that was quite an exercise in manipulation of sums, floors, and ceilings. Readers who are not accustomed to "proving that errors are small" might find it hard to believe that anybody would have the courage to keep going, when faced with such weird-looking sums. But actually, a second look shows that there's a simple motivating thread running through the whole calculation. The main idea is that a certain sum $s(\alpha, n, v)$ of n terms can be reduced to a similar sum of at most $\lceil n\alpha \rceil$ terms. Everything else cancels out except for a small residual left over from terms near the boundaries.

Let's take a deep breath now and do one more sum, which is not trivial but has the great advantage (compared with what we've just been doing) that

↓ Stop
↓ Skimming

it comes out in closed form so that we can easily check the answer. Our goal now will be to generalize the sum in (3.26) by finding an expression for

$$\sum_{0 \leq k < m} \left\lfloor \frac{nk + x}{m} \right\rfloor, \quad \text{integer } m > 0, \quad \text{integer } n.$$

Finding a closed form for this sum is tougher than what we've done so far (except perhaps for the discrepancy problem we just looked at). But it's instructive, so we'll hack away at it for the rest of this chapter.

As usual, especially with tough problems, we start by looking at small cases. The special case $n = 1$ is (3.26), with x replaced by x/m :

$$\left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{1+x}{m} \right\rfloor + \cdots + \left\lfloor \frac{m-1+x}{m} \right\rfloor = \lfloor x \rfloor.$$

And as in Chapter 1, we find it useful to get more data by generalizing downwards to the case $n = 0$:

$$\left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{x}{m} \right\rfloor + \cdots + \left\lfloor \frac{x}{m} \right\rfloor = m \left\lfloor \frac{x}{m} \right\rfloor.$$

Our problem has two parameters, m and n ; let's look at some small cases for m . When $m = 1$ there's just a single term in the sum and its value is $\lfloor x \rfloor$. When $m = 2$ the sum is $\lfloor x/2 \rfloor + \lfloor (x+n)/2 \rfloor$. We can remove the interaction between x and n by removing n from inside the floor function, but to do that we must consider even and odd n separately. If n is even, $n/2$ is an integer, so we can remove it from the floor:

$$\left\lfloor \frac{x}{2} \right\rfloor + \left(\left\lfloor \frac{x}{2} \right\rfloor + \frac{n}{2} \right) = 2 \left\lfloor \frac{x}{2} \right\rfloor + \frac{n}{2}.$$

If n is odd, $(n-1)/2$ is an integer so we get

$$\left\lfloor \frac{x}{2} \right\rfloor + \left(\left\lfloor \frac{x+1}{2} \right\rfloor + \frac{n-1}{2} \right) = \lfloor x \rfloor + \frac{n-1}{2}.$$

The last step follows from (3.26) with $m = 2$.

These formulas for even and odd n slightly resemble those for $n = 0$ and 1, but no clear pattern has emerged yet; so we had better continue exploring some more small cases. For $m = 3$ the sum is

$$\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x+n}{3} \right\rfloor + \left\lfloor \frac{x+2n}{3} \right\rfloor,$$

and we consider three cases for n : Either it's a multiple of 3, or it's 1 more than a multiple, or it's 2 more. That is, $n \bmod 3 = 0, 1, \text{ or } 2$. If $n \bmod 3 = 0$

Is this a harder sum of floors, or a sum of harder floors?

Be forewarned: This is the beginning of a pattern, in that the last part of the chapter consists of the solution of some long, difficult problem, with little more motivation than curiosity.

— Students

Touché. But c'mon, gang, do you always need to be told about applications before you can get interested in something? This sum arises, for example, in the study of random number generation and testing. But mathematicians looked at it long before computers came along, because they found it natural to ask if there's a way to sum arithmetic progressions that have been "flooded."

— Your instructor

then $n/3$ and $2n/3$ are integers, so the sum is

$$\left\lfloor \frac{x}{3} \right\rfloor + \left(\left\lfloor \frac{x}{3} \right\rfloor + \frac{n}{3} \right) + \left(\left\lfloor \frac{x}{3} \right\rfloor + \frac{2n}{3} \right) = 3 \left\lfloor \frac{x}{3} \right\rfloor + n.$$

If $n \bmod 3 = 1$ then $(n-1)/3$ and $(2n-2)/3$ are integers, so we have

$$\left\lfloor \frac{x}{3} \right\rfloor + \left(\left\lfloor \frac{x+1}{3} \right\rfloor + \frac{n-1}{3} \right) + \left(\left\lfloor \frac{x+2}{3} \right\rfloor + \frac{2n-2}{3} \right) = \lfloor x \rfloor + n - 1.$$

Again this last step follows from (3.26), this time with $m = 3$. And finally, if $n \bmod 3 = 2$ then

$$\left\lfloor \frac{x}{3} \right\rfloor + \left(\left\lfloor \frac{x+2}{3} \right\rfloor + \frac{n-2}{3} \right) + \left(\left\lfloor \frac{x+1}{3} \right\rfloor + \frac{2n-1}{3} \right) = \lfloor x \rfloor + n - 1.$$

“Inventive genius requires pleasurable mental activity as a condition for its vigorous exercise. ‘Necessity is the mother of invention’ is a silly proverb. ‘Necessity is the mother of futile dodges’ is much nearer to the truth. The basis of the growth of modern invention is science, and science is almost wholly the outgrowth of pleasurable intellectual curiosity.”

— A. N. Whitehead [371]

The left hemispheres of our brains have finished the case $m = 3$, but the right hemispheres still can't recognize the pattern, so we proceed to $m = 4$:

$$\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x+n}{4} \right\rfloor + \left\lfloor \frac{x+2n}{4} \right\rfloor + \left\lfloor \frac{x+3n}{4} \right\rfloor.$$

At least we know enough by now to consider cases based on $n \bmod m$. If $n \bmod 4 = 0$ then

$$\left\lfloor \frac{x}{4} \right\rfloor + \left(\left\lfloor \frac{x}{4} \right\rfloor + \frac{n}{4} \right) + \left(\left\lfloor \frac{x}{4} \right\rfloor + \frac{2n}{4} \right) + \left(\left\lfloor \frac{x}{4} \right\rfloor + \frac{3n}{4} \right) = 4 \left\lfloor \frac{x}{4} \right\rfloor + \frac{3n}{2}.$$

And if $n \bmod 4 = 1$,

$$\begin{aligned} \left\lfloor \frac{x}{4} \right\rfloor + \left(\left\lfloor \frac{x+1}{4} \right\rfloor + \frac{n-1}{4} \right) + \left(\left\lfloor \frac{x+2}{4} \right\rfloor + \frac{2n-2}{4} \right) + \left(\left\lfloor \frac{x+3}{4} \right\rfloor + \frac{3n-3}{4} \right) \\ = \lfloor x \rfloor + \frac{3n}{2} - \frac{3}{2}. \end{aligned}$$

The case $n \bmod 4 = 3$ turns out to give the same answer. Finally, in the case $n \bmod 4 = 2$ we get something a bit different, and this turns out to be an important clue to the behavior in general:

$$\begin{aligned} \left\lfloor \frac{x}{4} \right\rfloor + \left(\left\lfloor \frac{x+2}{4} \right\rfloor + \frac{n-2}{4} \right) + \left(\left\lfloor \frac{x}{4} \right\rfloor + \frac{2n}{4} \right) + \left(\left\lfloor \frac{x+2}{4} \right\rfloor + \frac{3n-2}{4} \right) \\ = 2 \left(\left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x+2}{4} \right\rfloor \right) + \frac{3n}{2} - 1 = 2 \left\lfloor \frac{x}{2} \right\rfloor + \frac{3n}{2} - 1. \end{aligned}$$

This last step simplifies something of the form $\lfloor y/2 \rfloor + \lfloor (y+1)/2 \rfloor$, which again is a special case of (3.26).

To summarize, here's the value of our sum for small m :

m	$n \bmod m = 0$	$n \bmod m = 1$	$n \bmod m = 2$	$n \bmod m = 3$
1	$\lfloor x \rfloor$			
2	$2 \lfloor \frac{x}{2} \rfloor + \frac{n}{2}$	$\lfloor x \rfloor + \frac{n}{2} - \frac{1}{2}$		
3	$3 \lfloor \frac{x}{3} \rfloor + n$	$\lfloor x \rfloor + n - 1$	$\lfloor x \rfloor + n - 1$	
4	$4 \lfloor \frac{x}{4} \rfloor + \frac{3n}{2}$	$\lfloor x \rfloor + \frac{3n}{2} - \frac{3}{2}$	$2 \lfloor \frac{x}{2} \rfloor + \frac{3n}{2} - 1$	$\lfloor x \rfloor + \frac{3n}{2} - \frac{3}{2}$

It looks as if we're getting something of the form

$$a \left\lfloor \frac{x}{a} \right\rfloor + bn + c,$$

where a , b , and c somehow depend on m and n . Even the myopic among us can see that b is probably $(m-1)/2$. It's harder to discern an expression for a ; but the case $n \bmod 4 = 2$ gives us a hint that a is probably $\gcd(m, n)$, the greatest common divisor of m and n . This makes sense because $\gcd(m, n)$ is the factor we remove from m and n when reducing the fraction n/m to lowest terms, and our sum involves the fraction n/m . (We'll look carefully at gcd operations in Chapter 4.) The value of c seems more mysterious, but perhaps it will drop out of our proofs for a and b .

In computing the sum for small m , we've effectively rewritten each term of the sum as

$$\left\lfloor \frac{x + kn}{m} \right\rfloor = \left\lfloor \frac{x + kn \bmod m}{m} \right\rfloor + \frac{kn}{m} - \frac{kn \bmod m}{m},$$

because $(kn - kn \bmod m)/m$ is an integer that can be removed from inside the floor brackets. Thus the original sum can be expanded into the following tableau:

$$\begin{array}{r}
 \left\lfloor \frac{x}{m} \right\rfloor \\
 + \left\lfloor \frac{x + n \bmod m}{m} \right\rfloor \\
 + \left\lfloor \frac{x + 2n \bmod m}{m} \right\rfloor \\
 \vdots \\
 + \left\lfloor \frac{x + (m-1)n \bmod m}{m} \right\rfloor
 \end{array}
 +
 \begin{array}{r}
 \frac{0}{m} \\
 \frac{n}{m} \\
 \frac{2n}{m} \\
 \vdots \\
 \frac{(m-1)n}{m}
 \end{array}
 -
 \begin{array}{r}
 \frac{0 \bmod m}{m} \\
 \frac{n \bmod m}{m} \\
 \frac{2n \bmod m}{m} \\
 \vdots \\
 \frac{(m-1)n \bmod m}{m}
 \end{array}.$$

When we experimented with small values of m , these three columns led respectively to $a\lfloor x/a \rfloor$, bn , and c .

In particular, we can see how b arises. The second column is an arithmetic progression, whose sum we know — it's the average of the first and last terms, times the number of terms:

$$\frac{1}{2} \left(0 + \frac{(m-1)n}{m} \right) \cdot m = \frac{(m-1)n}{2}.$$

So our guess that $b = (m-1)/2$ has been verified.

The first and third columns seem tougher; to determine a and c we must take a closer look at the sequence of numbers

$$0 \bmod m, \quad n \bmod m, \quad 2n \bmod m, \quad \dots, \quad (m-1)n \bmod m.$$

Suppose, for example, that $m = 12$ and $n = 5$. If we think of the sequence as times on a clock, the numbers are 0 o'clock (we take 12 o'clock to be 0 o'clock), then 5 o'clock, 10 o'clock, 3 o'clock (= 15 o'clock), 8 o'clock, and so on. It turns out that we hit every hour exactly once.

Now suppose $m = 12$ and $n = 8$. The numbers are 0 o'clock, 8 o'clock, 4 o'clock (= 16 o'clock), but then 0, 8, and 4 repeat. Since both 8 and 12 are multiples of 4, and since the numbers start at 0 (also a multiple of 4), there's no way to break out of this pattern — they must all be multiples of 4.

In these two cases we have $\gcd(12, 5) = 1$ and $\gcd(12, 8) = 4$. The general rule, which we will prove next chapter, states that if $d = \gcd(m, n)$ then we get the numbers $0, d, 2d, \dots, m-d$ in some order, followed by $d-1$ more copies of the same sequence. For example, with $m = 12$ and $n = 8$ the pattern 0, 8, 4 occurs four times.

The first column of our sum now makes complete sense. It contains d copies of the terms $\lfloor x/m \rfloor, \lfloor (x+d)/m \rfloor, \dots, \lfloor (x+m-d)/m \rfloor$, in some order, so its sum is

$$\begin{aligned} & d \left(\left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{x+d}{m} \right\rfloor + \dots + \left\lfloor \frac{x+m-d}{m} \right\rfloor \right) \\ &= d \left(\left\lfloor \frac{x/d}{m/d} \right\rfloor + \left\lfloor \frac{x/d+1}{m/d} \right\rfloor + \dots + \left\lfloor \frac{x/d+m/d-1}{m/d} \right\rfloor \right) \\ &= d \left\lfloor \frac{x}{d} \right\rfloor. \end{aligned}$$

This last step is yet another application of (3.26). Our guess for a has been verified:

$$a = d = \gcd(m, n).$$

*Lemma now,
dilemma later.*

Also, as we guessed, we can now compute c , because the third column has become easy to fathom. It contains d copies of the arithmetic progression $0/m, d/m, 2d/m, \dots, (m-d)/m$, so its sum is

$$d \left(\frac{1}{2} \left(0 + \frac{m-d}{m} \right) \cdot \frac{m}{d} \right) = \frac{m-d}{2};$$

the third column is actually subtracted, not added, so we have

$$c = \frac{d-m}{2}.$$

End of mystery, end of quest. The desired closed form is

$$\sum_{0 \leq k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor = d \left\lfloor \frac{x}{d} \right\rfloor + \frac{m-1}{2}n + \frac{d-m}{2},$$

where $d = \gcd(m, n)$. As a check, we can make sure this works in the special cases $n = 0$ and $n = 1$ that we knew before: When $n = 0$ we get $d = \gcd(m, 0) = m$; the last two terms of the formula are zero so the formula properly gives $m \lfloor x/m \rfloor$. And for $n = 1$ we get $d = \gcd(m, 1) = 1$; the last two terms cancel nicely, and the sum is just $\lfloor x \rfloor$.

By manipulating the closed form a bit, we can actually make it symmetric in m and n :

$$\begin{aligned} \sum_{0 \leq k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor &= d \left\lfloor \frac{x}{d} \right\rfloor + \frac{m-1}{2}n + \frac{d-m}{2} \\ &= d \left\lfloor \frac{x}{d} \right\rfloor + \frac{(m-1)(n-1)}{2} + \frac{m-1}{2} + \frac{d-m}{2} \\ &= d \left\lfloor \frac{x}{d} \right\rfloor + \frac{(m-1)(n-1)}{2} + \frac{d-1}{2}. \end{aligned} \quad (3.32)$$

This is astonishing, because there's no algebraic reason to suspect that such a sum should be symmetrical. We have proved a "reciprocity law,"

Yup, I'm floored.

$$\sum_{0 \leq k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor = \sum_{0 \leq k < n} \left\lfloor \frac{mk+x}{n} \right\rfloor, \quad \text{integers } m, n > 0.$$

For example, if $m = 41$ and $n = 127$, the left sum has 41 terms and the right has 127; but they still come out equal, for all real x .

Exercises

Warmups

- When we analyzed the Josephus problem in Chapter 1, we represented an arbitrary positive integer n in the form $n = 2^m + l$, where $0 \leq l < 2^m$. Give explicit formulas for l and m as functions of n , using floor and/or ceiling brackets.
- What is a formula for the nearest integer to a given real number x ? In case of ties, when x is exactly halfway between two integers, give an expression that rounds (a) up—that is, to $\lceil x \rceil$; (b) down—that is, to $\lfloor x \rfloor$.
- Evaluate $\lfloor \lfloor m\alpha \rfloor n / \alpha \rfloor$, when m and n are positive integers and α is an irrational number greater than n .
- The text describes problems at levels 1 through 5. What is a level 0 problem? (This, by the way, is *not* a level 0 problem.)
- Find a necessary and sufficient condition that $\lfloor nx \rfloor = n \lfloor x \rfloor$, when n is a positive integer. (Your condition should involve $\{x\}$.)
- Can something interesting be said about $\lfloor f(x) \rfloor$ when $f(x)$ is a continuous, monotonically *decreasing* function that takes integer values only when x is an integer?
- Solve the recurrence

$$\begin{aligned} X_n &= n, & \text{for } 0 \leq n < m; \\ X_n &= X_{n-m} + 1, & \text{for } n \geq m. \end{aligned}$$

You know you're in college when the book doesn't tell you how to pronounce 'Dirichlet'.

- Prove the *Dirichlet box principle*: If n objects are put into m boxes, some box must contain $\geq \lceil n/m \rceil$ objects, and some box must contain $\leq \lfloor n/m \rfloor$.
- Egyptian mathematicians in 1800 B.C. represented rational numbers between 0 and 1 as sums of unit fractions $1/x_1 + \cdots + 1/x_k$, where the x 's were distinct positive integers. For example, they wrote $\frac{1}{3} + \frac{1}{15}$ instead of $\frac{2}{5}$. Prove that it is always possible to do this in a systematic way: If $0 < m/n < 1$, then

$$\frac{m}{n} = \frac{1}{q} + \left\{ \text{representation of } \frac{m}{n} - \frac{1}{q} \right\}, \quad q = \left\lceil \frac{n}{m} \right\rceil.$$

(This is *Fibonacci's algorithm*, due to Leonardo Fibonacci, A.D. 1202.)

Basics

10 Show that the expression

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lfloor \frac{2x+1}{4} \right\rfloor + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumstances does each case arise?

11 Give details of the proof alluded to in the text, that the open interval $(\alpha.. \beta)$ contains exactly $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$ integers when $\alpha < \beta$. Why does the case $\alpha = \beta$ have to be excluded in order to make the proof correct?

12 Prove that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor,$$

for all integers n and all positive integers m . [This identity gives us another way to convert ceilings to floors and vice versa, instead of using the reflective law (3.4).]

13 Let α and β be positive real numbers. Prove that $\text{Spec}(\alpha)$ and $\text{Spec}(\beta)$ partition the positive integers if and only if α and β are irrational and $1/\alpha + 1/\beta = 1$.

14 Prove or disprove:

$$(x \bmod ny) \bmod y = x \bmod y, \quad \text{integer } n.$$

15 Is there an identity analogous to (3.26) that uses ceilings instead of floors?

16 Prove that $n \bmod 2 = (1 - (-1)^n)/2$. Find and prove a similar expression for $n \bmod 3$ in the form $a + b\omega^n + c\omega^{2n}$, where ω is the complex number $(-1 + i\sqrt{3})/2$. *Hint:* $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

17 Evaluate the sum $\sum_{0 \leq k < m} \lfloor x + k/m \rfloor$ in the case $x \geq 0$ by substituting $\sum_j [1 \leq j \leq x + k/m]$ for $\lfloor x + k/m \rfloor$ and summing first on k . Does your answer agree with (3.26)?

18 Prove that the boundary-value error term S in (3.30) is at most $\lceil \alpha^{-1} \nu \rceil$. *Hint:* Show that small values of j are not involved.

Homework exercises

19 Find a necessary and sufficient condition on the real number $b > 1$ such that

$$\lfloor \log_b x \rfloor = \lfloor \log_b \lfloor x \rfloor \rfloor$$

for all real $x \geq 1$.

- 20 Find the sum of all multiples of x in the closed interval $[\alpha \dots \beta]$, when $x > 0$.
- 21 How many of the numbers 2^m , for $0 \leq m \leq M$, have leading digit 1 in decimal notation?
- 22 Evaluate the sums $S_n = \sum_{k \geq 1} \lfloor n/2^k + \frac{1}{2} \rfloor$ and $T_n = \sum_{k \geq 1} 2^k \lfloor n/2^k + \frac{1}{2} \rfloor^2$.
- 23 Show that the n th element of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, \dots$$

is $\lfloor \sqrt{2n} + \frac{1}{2} \rfloor$. (The sequence contains exactly m occurrences of m .)

- 24 Exercise 13 establishes an interesting relation between the two multisets $\text{Spec}(\alpha)$ and $\text{Spec}(\alpha/(\alpha - 1))$, when α is any irrational number > 1 , because $1/\alpha + (\alpha - 1)/\alpha = 1$. Find (and prove) an interesting relation between the two multisets $\text{Spec}(\alpha)$ and $\text{Spec}(\alpha/(\alpha + 1))$, when α is any positive real number.
- 25 Prove or disprove that the Knuth numbers, defined by (3.16), satisfy $K_n \geq n$ for all nonnegative n .
- 26 Show that the auxiliary Josephus numbers (3.20) satisfy

$$\left(\frac{q}{q-1}\right)^n \leq D_n^{(q)} \leq q \left(\frac{q}{q-1}\right)^n, \quad \text{for } n \geq 0.$$

- 27 Prove that infinitely many of the numbers $D_n^{(3)}$ defined by (3.20) are even, and that infinitely many are odd.
- 28 Solve the recurrence

$$\begin{aligned} a_0 &= 1; \\ a_n &= a_{n-1} + \lfloor \sqrt{a_{n-1}} \rfloor, \quad \text{for } n > 0. \end{aligned}$$

There's a discrepancy between this formula and (3.31).

- 29 Show that, in addition to (3.31), we have

$$D(\alpha, n) \geq D(\alpha', \lfloor \alpha n \rfloor) - \alpha^{-1} - 2.$$

- 30 Show that the recurrence

$$\begin{aligned} X_0 &= m, \\ X_n &= X_{n-1}^2 - 2, \quad \text{for } n > 0, \end{aligned}$$

has the solution $X_n = \lceil \alpha^{2^n} \rceil$, if m is an integer greater than 2, where $\alpha + \alpha^{-1} = m$ and $\alpha > 1$. For example, if $m = 3$ the solution is

$$X_n = \lceil \phi^{2^{n+1}} \rceil, \quad \phi = \frac{1 + \sqrt{5}}{2}, \quad \alpha = \phi^2.$$

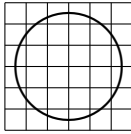
- 31 Prove or disprove: $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$.
- 32 Let $\|x\| = \min(x - \lfloor x \rfloor, \lceil x \rceil - x)$ denote the distance from x to the nearest integer. What is the value of

$$\sum_k 2^k \|x/2^k\|^2?$$

(Note that this sum can be doubly infinite. For example, when $x = 1/3$ the terms are nonzero as $k \rightarrow -\infty$ and also as $k \rightarrow +\infty$.)

Exam problems

- 33 A circle, $2n - 1$ units in diameter, has been drawn symmetrically on a $2n \times 2n$ chessboard, illustrated here for $n = 3$:



- a How many cells of the board contain a segment of the circle?
- b Find a function $f(n, k)$ such that exactly $\sum_{k=1}^{n-1} f(n, k)$ cells of the board lie entirely within the circle.
- 34 Let $f(n) = \sum_{k=1}^n \lceil \lg k \rceil$.
- a Find a closed form for $f(n)$, when $n \geq 1$.
- b Prove that $f(n) = n - 1 + f(\lceil n/2 \rceil) + f(\lfloor n/2 \rfloor)$ for all $n \geq 1$.
- 35 Simplify the formula $\lfloor (n+1)^2 n! e \rfloor \bmod n$.
- 36 Assuming that n is a nonnegative integer, find a closed form for the sum

$$\sum_{1 < k < 2^{2^n}} \frac{1}{2^{\lceil \lg k \rceil} 4^{\lfloor \lg k \rfloor}}.$$

Simplify it, but don't change the value.

- 37 Prove the identity

$$\sum_{0 \leq k < m} \left(\left\lfloor \frac{m+k}{n} \right\rfloor - \left\lfloor \frac{k}{n} \right\rfloor \right) = \left\lfloor \frac{m^2}{n} \right\rfloor - \left\lfloor \frac{\min(m \bmod n, (-m) \bmod n)^2}{n} \right\rfloor$$

for all positive integers m and n .

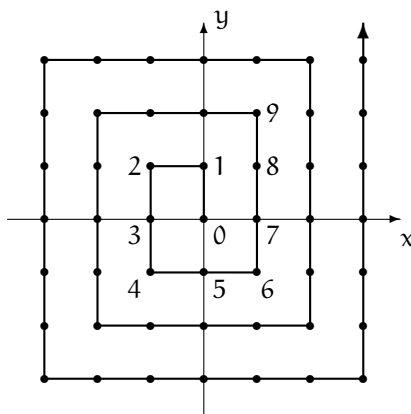
- 38 Let x_1, \dots, x_n be real numbers such that the identity

$$\sum_{k=1}^n \lfloor mx_k \rfloor = \left\lfloor m \sum_{1 \leq k \leq n} x_k \right\rfloor$$

holds for all positive integers m . Prove something interesting about x_1, \dots, x_n .

- 39 Prove that the double sum $\sum_{0 \leq k \leq \log_b x} \sum_{0 < j < b} \lceil (x + jb^k)/b^{k+1} \rceil$ equals $(b-1)(\lfloor \log_b x \rfloor + 1) + \lceil x \rceil - 1$, for every real number $x \geq 1$ and every integer $b > 1$.
- 40 The spiral function $\sigma(n)$, indicated in the diagram below, maps a non-negative integer n onto an ordered pair of integers $(x(n), y(n))$. For example, it maps $n = 9$ onto the ordered pair $(1, 2)$.

People in the southern hemisphere use a different spiral.



- a Prove that if $m = \lfloor \sqrt{n} \rfloor$,

$$x(n) = (-1)^m \left((n - m(m+1)) \cdot \lfloor 2\sqrt{n} \rfloor \text{ is even} \rfloor + \lfloor \frac{1}{2}m \rfloor \right),$$

and find a similar formula for $y(n)$. *Hint:* Classify the spiral into segments W_k, S_k, E_k, N_k according as $\lfloor 2\sqrt{n} \rfloor = 4k-2, 4k-1, 4k, 4k+1$.

- b Prove that, conversely, we can determine n from $\sigma(n)$ by a formula of the form

$$n = (2k)^2 \pm (2k + x(n) + y(n)), \quad k = \max(|x(n)|, |y(n)|).$$

Give a rule for when the sign is $+$ and when the sign is $-$.

Bonus problems

- 41 Let f and g be increasing functions such that the sets $\{f(1), f(2), \dots\}$ and $\{g(1), g(2), \dots\}$ partition the positive integers. Suppose that f and g are related by the condition $g(n) = f(f(n)) + 1$ for all $n > 0$. Prove that $f(n) = \lfloor n\phi \rfloor$ and $g(n) = \lfloor n\phi^2 \rfloor$, where $\phi = (1 + \sqrt{5})/2$.
- 42 Do there exist real numbers α, β , and γ such that $\text{Spec}(\alpha), \text{Spec}(\beta)$, and $\text{Spec}(\gamma)$ together partition the set of positive integers?

- 43 Find an interesting interpretation of the Knuth numbers, by unfolding the recurrence (3.16).
- 44 Show that there are integers $a_n^{(q)}$ and $d_n^{(q)}$ such that

$$a_n^{(q)} = \frac{D_{n-1}^{(q)} + d_n^{(q)}}{q-1} = \frac{D_n^{(q)} + d_n^{(q)}}{q}, \quad \text{for } n > 0,$$

when $D_n^{(q)}$ is the solution to (3.20). Use this fact to obtain another form of the solution to the generalized Josephus problem:

$$J_q(n) = 1 + d_k^{(q)} + q(n - a_k^{(q)}), \quad \text{for } a_k^{(q)} \leq n < a_{k+1}^{(q)}.$$

- 45 Extend the trick of exercise 30 to find a closed-form solution to

$$\begin{aligned} Y_0 &= m, \\ Y_n &= 2Y_{n-1}^2 - 1, \quad \text{for } n > 0, \end{aligned}$$

if m is a positive integer.

- 46 Prove that if $n = \lfloor (\sqrt{2}^l + \sqrt{2}^{l-1})m \rfloor$, where m and l are nonnegative integers, then $\lfloor \sqrt{2n(n+1)} \rfloor = \lfloor (\sqrt{2}^{l+1} + \sqrt{2}^l)m \rfloor$. Use this remarkable property to find a closed form solution to the recurrence

$$\begin{aligned} L_0 &= a, & \text{integer } a > 0; \\ L_n &= \lfloor \sqrt{2L_{n-1}(L_{n-1} + 1)} \rfloor, & \text{for } n > 0. \end{aligned}$$

Hint: $\lfloor \sqrt{2n(n+1)} \rfloor = \lfloor \sqrt{2}(n + \frac{1}{2}) \rfloor$.

- 47 The function $f(x)$ is said to be *replicative* if it satisfies

$$f(mx) = f(x) + f\left(x + \frac{1}{m}\right) + \cdots + f\left(x + \frac{m-1}{m}\right)$$

for every positive integer m . Find necessary and sufficient conditions on the real number c for the following functions to be replicative:

- a $f(x) = x + c$.
- b $f(x) = [x + c \text{ is an integer}]$.
- c $f(x) = \max([x], c)$.
- d $f(x) = x + c[x] - \frac{1}{2}[x \text{ is not an integer}]$.

- 48 Prove the identity

$$x^3 = 3x[x[x]] + 3\{x\}\{x[x]\} + \{x\}^3 - 3[x][x[x]] + [x]^3,$$

and show how to obtain similar formulas for x^n when $n > 3$.

- 49 Find a necessary and sufficient condition on the real numbers $0 \leq \alpha < 1$ and $\beta \geq 0$ such that we can determine α and β from the infinite multiset of values

$$\{ \lfloor n\alpha \rfloor + \lfloor n\beta \rfloor \mid n > 0 \}.$$

Research problems

- 50 Find a necessary and sufficient condition on the nonnegative real numbers α and β such that we can determine α and β from the infinite multiset of values

$$\{ \lfloor \lfloor n\alpha \rfloor \beta \rfloor \mid n > 0 \}.$$

- 51 Let x be a real number $\geq \phi = \frac{1}{2}(1 + \sqrt{5})$. The solution to the recurrence

$$\begin{aligned} Z_0(x) &= x, \\ Z_n(x) &= Z_{n-1}(x)^2 - 1, \quad \text{for } n > 0, \end{aligned}$$

can be written $Z_n(x) = \lceil f(x)^{2^n} \rceil$, if x is an integer, where

$$f(x) = \lim_{n \rightarrow \infty} Z_n(x)^{1/2^n},$$

because $Z_n(x) - 1 < f(x)^{2^n} < Z_n(x)$ in that case. What other interesting properties does this function $f(x)$ have?

- 52 Given nonnegative real numbers α and β , let

$$\text{Spec}(\alpha; \beta) = \{ \lfloor \alpha + \beta \rfloor, \lfloor 2\alpha + \beta \rfloor, \lfloor 3\alpha + \beta \rfloor, \dots \}$$

Spec this be hard.

be a multiset that generalizes $\text{Spec}(\alpha) = \text{Spec}(\alpha; 0)$. Prove or disprove: If the $m \geq 3$ multisets $\text{Spec}(\alpha_1; \beta_1)$, $\text{Spec}(\alpha_2; \beta_2)$, \dots , $\text{Spec}(\alpha_m; \beta_m)$ partition the positive integers, and if the parameters $\alpha_1 < \alpha_2 < \dots < \alpha_m$ are rational, then

$$\alpha_k = \frac{2^m - 1}{2^{k-1}}, \quad \text{for } 1 \leq k \leq m.$$

- 53 Fibonacci's algorithm (exercise 9) is "greedy" in the sense that it chooses the least conceivable q at every step. A more complicated algorithm is known by which every fraction m/n with n odd can be represented as a sum of distinct unit fractions $1/q_1 + \dots + 1/q_k$ with *odd* denominators. Does the greedy algorithm for such a representation always terminate?

Index

(*Graffiti have been indexed too.*)

WHEN AN INDEX ENTRY refers to a page containing a relevant exercise, the answer to that exercise (in Appendix A) might divulge further information; an answer page is not indexed here unless it refers to a topic that isn't included in the statement of the relevant exercise. Some notations not indexed here (like $x^{\mathbb{N}}$, $\lfloor x \rfloor$, and $\langle \binom{n}{m} \rangle$) are listed on pages x and xi , just before the table of contents.

0^0 , 162
 $\sqrt{2}$ (≈ 1.41421), 100
 $\sqrt{3}$ (≈ 1.73205), 378
 \Im : imaginary part, 64
 \mathcal{L} : logarithmico-exponential functions, 442–443
 \Re : real part, 64, 212, 451
 γ (≈ 0.57722), *see* Euler's constant
 Γ , *see* Gamma function
 δ , 47–56
 Δ : difference operator, 47–55, 241, 470–471
 $e_p(n)$: largest power of p dividing n , 112–114, 146
 ζ , *see* zeta function
 ϑ , 219–221, 310, 347
 Θ : Big Theta notation, 448
 κ_m , *see* cumulants
 μ , *see* Möbius function
 ν , *see* nu function
 π (≈ 3.14159), 26, 70, 146, 244, 485, 564, 596
 $\pi(x)$, *see* pi function
 σ : standard deviation, 388; *see also* Stirling's constant
 $\sigma_n(x)$, *see* Stirling polynomials
 ϕ (≈ 1.61803): golden ratio, 70, 97, 299–301, 310, 553
 φ , *see* phi function
 Φ : sum of φ , 138–139, 462–463
 Ω : Big Omega notation, 448
 Σ -notation, 22–25, 245

\prod -notation, 64, 106
 \wedge -notation, 65
 \iff : if and only if, 68
 \implies : implies, 71
 \backslash : divides, 102
 \parallel : exactly divides, 146
 \perp : is relatively prime to, 115
 \prec : grows slower than, 440–443
 \succ : grows faster than, 440–443
 \asymp : grows as fast as, 442–443
 \sim : is asymptotic to, 8, 110, 439–443, 448–449
 \approx : approximates, 23
 \equiv : is congruent to, 123–126
 $\#$: cardinality, 39
 $!$: factorial, 111–115
 j : subfactorial, 194–200
 \dots : interval notation, 73–74
 \dots : ellipsis, 21, 50, 108, ...
Aaronson, Bette Jane, ix
Abel, Niels Henrik, 604, 634
Abramowitz, Milton, 42, 604
absolute convergence, 60–62, 64
absolute error, 452, 455
absolute value of complex number, 64
absorption identities, 157–158, 261
Acton, John Emerich Edward Dalberg, Baron, 66
Adams, William Wells, 604, 635
Addison-Wesley, ix

- addition formula for $\binom{n}{k}$, 158–159
 analog for $\langle \binom{n}{k} \rangle$, 268
 analogs for $\{\binom{n}{k}\}$ and $[\binom{n}{k}]$, 259, 261
 dual, 530
- Aho, Alfred Vaino, 604, 633
- Ahrens, Wilhelm Ernst Martin Georg, 8, 604
- Akhiezer, Naum Il'ich, 604
- Alfred [Brousseau], Brother Ulbertus, 607, 633
- algebraic integers, 106, 147
- algorithms,
 analysis of, 138, 413–426
 divide and conquer, 79
 Euclid's, 103–104, 123, 303–304
 Fibonacci's, 95, 101
 Gosper's, 224–227
 Gosper–Zeilberger, 229–241, 254–255, 319, 547
 greedy, 101, 295
 self-certifying, 104
- Alice, 31, 408–410, 427, 430
- Allardice, Robert Edgar, 2, 604
- ambiguous notation, 245
- American Mathematical Society, viii
- AMS Euler, ix, 657
- analysis of algorithms, 138, 413–426
- analytic functions, 196
- ancestor, 117, 291
- André, Antoine Désiré, 604, 635
- Andrews, George W. Eyre, 215, 330, 530, 575, 605, 634, 635
- answers, notes on, 497, 637, viii
- anti-derivative operator, 48, 470–471
- anti-difference operator, 48, 54, 470–471
- Apéry, Roger, 238, 605, 630, 634
 numbers, 238–239, 255
- approximation, *see* asymptotics
 of sums by integrals, 45, 276–277, 469–475
- Archibald, Raymond Clare, 608
- Archimedes of Syracuse, 6
- argument of hypergeometric, 205
- arithmetic progression, 30, 376
 floored, 89–94
 sum of, 6, 26, 30–31
- Armageddon, 85
- Armstrong, Daniel Louis (= Satchmo), 80
- art and science, 234
- ascents, 267–268, 270
- Askey, Richard Allen, 634
- associative law, 30, 61, 64
- asymptotics, 439–496
 from convergent series, 451
 of Bernoulli numbers, 286, 452
 of binomial coefficients, 248, 251, 495, 598
 of discrepancies, 492, 495
 of factorials, 112, 452, 481–482, 491
 of harmonic numbers, 276–278, 452, 480–481, 491
 of hashing, 426
 of n th prime, 110–111, 456–457, 490
 of Stirling numbers, 495, 602
 of sums, using Euler's summation formula, 469–489
 of sums, using tail-exchange, 466–469, 486–489
 of sums of powers, 491
 of wheel winners, 76, 453–454
 table of expansions, 452
 usefulness of, 76, 439
- Atkinson, Michael David, 605, 633, 635
- Austin, Alan Keith, 607
- automaton, 405
- automorphic numbers, 520
- average, 384
 of a reciprocal, 432
 variance, 423–425
- B_n , *see* Bernoulli numbers
- Bachmann, Paul Gustav Heinrich, 443, 462, 605
- Bailey, Wilfrid Norman, 223, 548, 605, 634
- Balasubramanian, Ramachandran, 525, 605.
- Ball, Walter William Rouse, 605, 633
- ballot problem, 362
- Banach, Stefan, 433
- Barlow, Peter, 605, 634
- Barton, David Elliott, 602, 609
- base term, 240
- baseball, 73, 148, 195, 519, 622, 648, 653
- BASIC, 173, 446
- basic fractions, 134, 138
- basis of induction, 3, 10–11, 320–321
- Bateman, Harry, 626
- Baum, Lyman Frank, 581
- Beatty, Samuel, 605, 633
- bee trees, 291
- Beeton, Barbara Ann Neuhaus Friend Smith, viii
- Bell, Eric Temple, 332, 606, 635
 numbers, 373, 493, 603
- Bender, Edward Anton, 606, 636

- Bernoulli, Daniel, 299
- Bernoulli, Jakob (= Jacobi = Jacques = James), 283, 470, 606
 numbers, *see* Bernoulli numbers
 polynomials, 367–368, 470–475
 polynomials, graphs of, 473
 trials, 402; *see also* coins, flipping
- Bernoulli, Johann (= Jean), 622
- Bernoulli numbers, 283–290
 asymptotics of, 286, 452
 calculation of, 288, 620
 denominators of, 315, 551, 574
 generalized, *see* Stirling polynomials
 generating function for, 285, 351, 365
 numerators of, 555
 relation to tangent numbers, 287
 table of, 284, 620
- Bernshtein (= Bernstein), Sergeĭ Natanovich, 636
- Bertrand, Joseph Louis François, 145, 606, 633
 postulate, 145, 500, 550
- Bessel, Friedrich Wilhelm, functions, 206, 527
- Beyer, William Hyman, 606
- biased coin, 401
- bicycles, 260, 500
- Bieberbach, Ludwig, 617
- Bienaymé, Irénée Jules, 606
- Big Ell notation, 444
- Big Oh notation, 76, 443–449
- Big Omega notation, 448
- Big Theta notation, 448
- bijection, 39
- Bill, 408–410, 427, 430
- binary logarithm, 70, 449
- binary notation (radix 2), 11–13, 15–16, 70, 113–114
- binary partitions, 377
- binary search, 121, 183
- binary trees, 117
- Binet, Jacques Philippe Marie, 299, 303, 606, 633
- binomial coefficients, 153–242
 addition formula, 158–159
 asymptotics of, 248, 251, 495, 598
 combinatorial interpretation, 153, 158, 160, 169–170
 definition, 154, 211
 dual, 530
 fractional, 250
 generalized, 211, 318, 530
 indices of, 154
 middle, 187, 255–256, 495
 reciprocal of, 188–189, 246, 254
 top ten identities of, 174
 wraparound, 250 (exercise 75), 315
- binomial convolution, 365, 367
- binomial distribution, 401–402, 415, 428, 432
 negative, 402–403, 428
- binomial series, generalized, 200–204, 243, 252, 363
- binomial theorem, 162–163
 as hypergeometric series, 206, 221
 discovered mechanically, 230–233
 for factorial powers, 245
 special cases, 163, 199
- Blom, Carl Gunnar, 606, 636
- bloopergeometric series, 243
- Boas, Ralph Philip, Jr., 600, 606, 636, viii
- Boggs, Wade Anthony, 195
- Bohl, Piers Paul Felix (= Bol', Pirs Georgievich), 87, 606
- Böhmer, Paul Eugen, 604
- Bois-Reymond, Paul David Gustav du, 440, 610, 617
- Boncompagni, Prince Baldassarre, 613
- bootstrapping, 463–466
 to estimate n th prime, 456–457
- Borchardt, Carl Wilhelm, 617
- Borel, Émile Félix Édouard Justin, 606, 636
- Borwein, Jonathan Michael, 606, 635
- Borwein, Peter Benjamin, 606, 635
- bound variables, 22
- boundary conditions on sums,
 can be difficult, 75, 86
 made easier, 24–25, 159
- bowling, 6
- box principle, 95, 130, 512
- Boyd, David William, 564
- bracket notation,
 for coefficients, 197, 331
 for true/false values, 24–25
- Brahma, Tower of, 1, 4, 278
- Branges, Louis de, 617
- Brent, Richard Peirce, 306, 525, 564, 606
- bricks, 313, 374
- Brillhart, John David, 607, 633
- Brocot, Achille, 116, 607
- Broder, Andrei Zary, 632, ix
- Brooke, Maxey, 607, 635

- Brousseau, Brother Alfred, 607, 633
 Brown, Mark Robbin, 632
 Brown, Morton, 501, 607
 Brown, Roy Howard, ix
 Brown, Thomas Craig, 607, 633
 Brown, Trivial, 607
 Brown, William Gordon, 607
 Brown University, ix
 Browning, Elizabeth Barrett, 320
 Bruijn, Nicolaas Govert de, 444, 447, 500, 609, 635, 636
 cycle, 500
 bubblesort, 448
 Buckholtz, Thomas Joel, 620
 Bulwer-Lytton, Edward George Earle Lytton, Baron, v
 Burma-Shave, 541
 Burr, Stefan Andrus, 607, 635

 calculators, 67, 77, 459
 failure of, 344
 calculus, vi, 33
 finite and infinite, 47–56
 candy, 36
 Canfield, Earl Rodney, 602, 607, 636
 cards,
 shuffling, 437
 stacking, 273–274, 278, 309
 Carlitz, Leonard, 607, 635
 Carroll, Lewis (= Dodgson, Rev. Charles Lutwidge), 31, 293, 607, 608, 630
 carries,
 across the decimal point, 70
 in divisibility of $\binom{m+n}{m}$, 245, 536
 in Fibonacci number system, 297, 561
 Cassini, Gian (= Giovanni = Jean) Domenico (= Dominique), 292, 607
 identity, 292–293, 300
 identity, converse, 314
 identity, generalized, 303, 310
 Catalan, Eugène Charles, 203, 361, 607
 Catalan numbers, 203
 combinatorial interpretations, 358–360, 565, 568
 generalized, 361
 in sums, 181, 203, 317
 table of, 203
 Cauchy, Augustin Louis, 607, 633
 Čech, Eduard, vi

 ceiling function, 67–69
 converted to floor, 68, 96
 graph of, 68
 center of gravity, 273–274, 309
 certificate of correctness, 104
 Chace, Arnold Buffum, 608, 633
 Chaimovich, Mark, 608
 chain rule, 54, 483
 change, 327–330, 374
 large amounts of, 344–346, 492
 changing the index of summation, 30–31, 39
 changing the tails of a sum, 466–469, 486–489
 cheating, viii, 195, 388, 401
 not, 158, 323
 Chebyshev, Pafnutii L'vovich, 38, 145, 608, 633
 inequality, 390–391, 428, 430
 monotonic inequalities, 38, 576
 cheese slicing, 19
 Chen, Pang-Chieh, 632
 Chinese Remainder Theorem, 126, 146
 Chu Shih-Chieh (= Zhū Shijié), 169
 Chung, Fan-Rong King, ix, 608, 635
 Clausen, Thomas, 608, 634, 635
 product identities, 253
 clearly, clarified, 417–418, 581
 clichés, 166, 324, 357
 closed form, 3, 7, 321, 331
 for generating functions, 317
 not, 108, 573
 pretty good, 346
 closed interval, 73–74
 Cobb, Tyrus Raymond, 195
 coefficient extraction, 197, 331
 Cohen, Henri José, 238
 coins, 327–330
 biased, 401
 fair, 401, 430
 flipping, 401–410, 430–432, 437–438
 spinning, 401
 Collingwood, Stuart Dodgson, 608
 Collins, John, 624
 Colombo, Cristoforo (= Columbus, Christopher), 74
 coloring, 496
 Columbia University, ix
 combinations, 153
 combinatorial number system, 245
 common logarithm, 449

- commutative law, 30, 61, 64
 - failure of, 322, 502, 551
 - relaxed, 31
- complete graph, 368
- complex factorial powers, 211
- complex numbers, 64
 - roots of unity, 149, 204, 375, 553, 574, 598
- composite numbers, 105, 518
- composition of generating functions, 428
- computer algebra, 42, 254, 501, 539
- Comtet, Louis, 609, 636
- Concrete Math Club, 74, 453
- concrete mathematics, defined, vi
- conditional convergence, 59
- conditional probability, 416–419, 424–425
- confluent hypergeometric series, 206, 245
- congruences, 123–126
- Connection Machine, 131
- contiguous hypergeometrics, 529
- continuants, 301–309, 501
 - and matrices, 318–319
 - Euler's identity for, 303, 312
 - zero parameters in, 314
- continued fractions, 301, 304–309, 319
 - large partial quotients of, 553, 563, 564, 602
- convergence,
 - absolute, 60–62, 64
 - conditional, 59
 - of power series, 206, 331–332, 348, 451, 532
- convex regions, 5, 20, 497
- convolution, 197, 246, 333, 353–364
 - binomial, 365, 367
 - identities for, 202, 272, 373
 - polynomials, 373
 - Stirling, 272, 290
 - Vandermonde, *see* Vandermonde convolution
- Conway, John Horton, 410, 609
- cotangent function, 286, 317
- counting,
 - combinations, 153
 - cycle arrangements, 259–262
 - derangements, 193–196, 199–200
 - integers in intervals, 73–74
 - necklaces, 139–141
 - parenthesized formulas, 357–359
 - permutations, 111
 - permutations by ascents, 267–268
 - permutations by cycles, 262
 - set partitions, 258–259
 - spanning trees, 348–350, 356, 368–369, 374
 - with generating functions, 320–330
- coupon collecting, 583
- Cover, Thomas Merrill, 636
- Coxeter, Harold Scott MacDonald, 605
- Cramér, Carl Harald, 525, 609, 634
- Cray X-MP, 109
- Crelle, August Leopold, 609, 633
- cribbage, 65
- Crispin, Mark Reed, 628
- Crowe, Donald Warren, 609, 633
- crudification, 447
- Csirik, János András, 590, 609
- cubes, sum of consecutive, 51, 63, 283, 289, 367
- cumulants, 397–401
 - infinite, 576
 - of binomial distribution, 432
 - of discrete distribution, 438
 - of Poisson distribution, 428–429
 - third and fourth, 429, 579, 589
- CUNY (= City University of New York), ix
- Curtiss, David Raymond, 609, 634
- cycles,
 - de Bruijn, 500
 - of beads, 139–140
 - of permutations, 259–262
- cyclic shift, 12, 359, 362
- cyclotomic polynomials, 149
- D, *see* derivative operator
- Dating Game, 506
- David, Florence Nightingale, 602, 609
- Davis, Philip Jacob, 609
- Davison, John Leslie, 307, 604, 609, 635
- de Branges, Louis, 617
- de Bruijn, Nicolaas Govert, 444, 447, 500, 609, 635, 636
 - cycle, 500
- de Finetti, Bruno, 24, 613
- de Lagny, Thomas Fantet, 304, 621
- de Moivre, Abraham, 297, 481, 609
- Dedekind, Julius Wilhelm Richard, 136–137, 609
- definite sums, analogous to definite integrals, 49–50
- deg, 227, 232
- degenerate hypergeometric series, 209–210, 216, 222, 247
- derangements, 194–196, 250
 - generating function, 199–200
- derivative operator, 47–49
 - converting between D and Δ , 470–471
 - converting between D and ∂ , 310
 - with generating functions, 33, 333, 364–365
 - with hypergeometric series, 219–221

- descents, *see* ascents
- dgf: Dirichlet generating function, 370
- dice, 381–384
 - fair, 382, 417, 429
 - loaded, 382, 429, 431
 - nonstandard, 431
 - pgf for, 399–400
 - probability of doubles, 427
 - supposedly fair, 392
- Dickson, Leonard Eugene, 510, 609
- Dieudonné, Jean Alexandre, 523
- difference operator, 47–55, 241
 - converting between D and Δ , 470–471
 - n th difference, 187–192, 280–281
 - n th difference of product, 571
- differentially finite power series, 374, 380
- differential operators, *see* derivative operator, theta operator
- difficulty measure for summation, 181
- Dijkstra, Edsger Wybe, 173, 609, 635
- dimers and dimes, 320, *see* dominoes and change
- diphages, 434, 438
- Dirichlet, Johann Peter Gustav Lejeune, 370, 610, 633
 - box principle, 95, 130, 512
 - generating functions, 370–371, 373, 432, 451
 - probability generating functions, 432
- discrepancy, 88–89, 97
 - and continued fractions, 319, 492, 602
 - asymptotics of, 492, 495
- discrete probability, 381–438
 - defined, 381
- distribution,
 - of fractional parts, 87
 - of primes, 111
 - of probabilities, *see* probability distributions
 - of things into groups, 83–85
- distributive law, 30, 35, 60, 64
 - for gcd and lcm, 145
 - for mod, 83
- divergent sums, 57, 60
 - considered useful, 346–348, 451
 - illegitimate, 504, 532
- divide and conquer, 79
- divides exactly, 146
 - in binomial coefficients, 245
 - in factorials, 112–114, 146
- divisibility, 102–105
 - by 3, 147
 - of polynomials, 225
- Dixon, Alfred Cardew, 610, 634
 - formula, 214
- DNA, Martian, 377
- Dodgson, Charles Lutwidge, *see* Carroll
- domino tilings, 320–327, 371, 379
 - ordered pairs of, 375
- Dorothy Gale, 581
- double generating functions, *see* super generating functions
- double sums, 34–41, 246, 249
 - considered useful, 46, 183–185
 - faulty use of, 63, 65
 - infinite, 61
 - over divisors, 105
 - telescoping, 255
- doubloons, 436–437
- doubly exponential recurrences, 97, 100, 101, 109
- doubly infinite sums, 59, 98, 482–483
- Dougall, John, 171, 610
- downward generalization, 2, 95, 320–321
- Doyle, Sir Arthur Ignatius Conan, 162, 228–229, 405, 610
- drones, 291
- Drysdale, Robert Lewis (Scot), III, 632
- du Bois-Reymond, Paul David Gustav, 440, 610, 617
- duality, 69
 - between $\binom{n}{k}$ and $1/n\binom{n-1}{k}$, 530
 - between factorial and Gamma functions, 211
 - between floors and ceilings, 68–69, 96
 - between gcd and lcm, 107
 - between rising and falling powers, 63
 - between Stirling numbers of different kinds, 267
- Dubner, Harvey, 610, 631, 633
- Dudeney, Henry Ernest, 610, 633
- Dunkel, Otto, 614, 633
- Dunn, Angela Fox, 628, 635
- Dunnington, Guy Waldo, 610
- duplication formulas, 186, 244
- Dupré, Lyn Oppenheim, ix
- Durst, Lincoln Kearney, viii
- Dyson, Freeman John, 172, 239, 610, 615
- e ($\approx 2.718281828459045$),
 - as canonical constant, 70, 596
 - representations of, 122, 150
- e_n , *see* Euclid numbers
- E: expected value, 385–386

- E: shift operator, 55, 188, 191
- E_n , see Euler numbers
- Edwards, Anthony William Fairbank, 610
- eeny-meeny-miny-mo, see Josephus problem
- efficiency, different notions of, 24, 133
- egf: exponential generating function, 364
- eggs, 158
- Egyptian mathematics, 95, 150
 - bibliography of, 608
- Einstein, Albert, 72, 307
- Eisele, Carolyn, 625
- Eisenstein, Ferdinand Gotthold Max, 202, 610
- Ekhad, Shalosh B, 546
- elementary events, 381–382
- Elkies, Noam David, 131, 610
- ellipsis (\dots), 21
 - advantage of, 21, 25, 50
 - disadvantage of, 25
 - elimination of, 108
- empirical estimates, 391–393, 427
- empty case,
 - for spanning trees, 349, 565
 - for Stirling numbers, 258
 - for tilings, 320–321
 - for Tower of Hanoi, 2
- empty product, 48, 106, 111
- empty sum, 24, 48
- entier function, see floor function
- equality, one-way, 446–447, 489–490
- equivalence relation, 124
- Eratosthenes, sieve of, 111
- Erdélyi, Arthur, 629, 636
- Erdős, Pál (= Paul), 418, 525, 548, 575, 610–611, 634, 636
- error function, 166
- errors, absolute versus relative, 452, 455
- errors, locating our own, 183
- Eswarathasan, Arulappah, 611, 635
- Euclid (= *Εὐκλείδης*), 107–108, 147, 611
 - algorithm, 103–104, 123, 303–304
 - numbers, 108–109, 145, 147, 150, 151
- Euler, Leonhard, i, vii, ix, 48, 122, 132–134, 202, 205, 207, 210, 267, 277, 278, 286, 301–303, 469, 471, 513, 529, 551, 575, 603, 605, 609, 611–613, 629, 630, 633–636
 - constant (≈ 0.57722), 278, 306–307, 316, 319, 481, 596
 - disproved conjecture, 131
 - identity for continuants, 303, 312
 - identity for hypergeometrics, 244
 - numbers, 559, 570, 620; see also Eulerian numbers
 - polynomials, 574
 - pronunciation of name, 147
 - summation formula, 469–475
 - theorem, 133, 142, 147
 - totient function, see phi function
 - triangle, 268, 316
- Eulerian numbers, 267–271, 310, 316, 378, 574
 - combinatorial interpretations, 267–268, 557
 - generalized, 313
 - generating function for, 351
 - second-order, 270–271
 - table of, 268
- event, 382
- eventually positive function, 442
- exact cover, 376
- exactly divides, 146
 - in binomial coefficients, 245
 - in factorials, 112–114, 146
- excedances, 316
- exercises, levels of, viii, 72–73, 95, 511
- exp: exponential function, 455
- expectation, see expected value
- expected value, 385–387
 - using a pgf, 395
- exponential function, discrete analog of, 54
- exponential generating functions, 364–369, 421–422
- exponential series, generalized, 200–202, 242, 364, 369
- exponents, laws of, 52, 63
- F, see hypergeometric series
- F_n , see Fibonacci numbers
- factorial expansion of binomial coefficients, 156, 211
- factorial function, 111–115, 346–348
 - approximation to, see Stirling's approximation
 - duplication formula, 244
 - generalized to nonintegers, 192, 210–211, 213–214, 316
- factorial powers, see falling factorial powers, rising factorial powers
- factorization into primes, 106–107, 110
- factorization of summation conditions, 36
- fair coins, 401, 430
- fair dice, 382, 386, 392, 417, 429

- falling factorial powers, 47
 - binomial theorem for, 245
 - complex, 211
 - difference of, 48, 53, 188
 - negative, 52, 63, 188
 - related to ordinary powers, 51, 262–263, 598
 - related to rising powers, 63, 312
 - summation of, 50–53
- fans, ix, 193, 348
- Farey, John, series, 118–119, 617
 - consecutive elements of, 118–119, 150
 - distribution of, 152
 - enumeration of, 134, 137–139, 462–463
- Fasenmyer, Mary Celine, 230, 631
- Faulhaber, Johann, 288, 613, 620
- Feder Bermann, Tomás, 635
- Feigenbaum, Joan, 632
- Feller, William, 381, 613, 636
- Fermat, Pierre de, 130, 131, 613
 - numbers, 131–132, 145, 525
- Fermat's Last Theorem, 130–131, 150, 524, 555
- Fermat's theorem (= Fermat's Little Theorem), 131–133, 141–143, 149
 - converse of, 132, 148
- Fibonacci, Leonardo, of Pisa (= Leonardo filio Bonacci Pisano), 95, 292, 549, 613, 633, 634
 - addition, 296–297, 318
 - algorithm, 95, 101
 - factorial, 492
 - multiplication, 561
 - number system, 296–297, 301, 307, 310, 318
 - odd and even, 307–308
- Fibonacci numbers, 290–301, 575
 - and continuants, 302
 - and sunflowers, 291
 - closed forms for, 299–300, 331
 - combinatorial interpretations of, 291–292, 302, 321, 549
 - egf for, 570
 - ordinary generating functions for, 297–300, 337–340, 351
 - second-order, 375
 - table of, 290, 293
- Fibonomial coefficients, 318, 556
- Fine, Henry Burchard, 625
- Fine, Nathan Jacob, 603
- Finetti, Bruno de, 24, 613
- finite calculus, 47–56
- finite state language, 405
- Finkel, Raphael Ari, 628
- Fisher, Michael Ellis, 613, 636
- Fisher, Sir Ronald Aylmer, 613, 636
- fixed points, 12, 393–394
 - pgf for, 400–401, 428
- Flajolet, Philippe Patrick Michel, 564
- flipping coins, 401–410, 430–432, 437–438
- floor function, 67–69
 - converted to ceiling, 68, 96
 - graph of, 68
- Floyd, Robert W, 635
- food, *see* candy, cheese, eggs, pizza, sherry
- football, 182
- football victory problem, 193–196, 199–200, 428
 - generalized, 429
 - mean and variance, 393–394, 400–401
- Forcadel, Pierre, 613, 634
- formal power series, 206, 331, 348, 532
- FORTRAN, 446
- Fourier, Jean Baptiste Joseph, 22, 613
 - series, 495
- fractional parts, 70
 - in Euler's summation formula, 470
 - in polynomials, 100
 - related to mod, 83
 - uniformly distributed, 87
- fractions, 116–123
 - basic, 134, 138
 - continued, 301, 304–309, 319, 564
 - partial, *see* partial fraction expansions
 - unit, 95, 101, 150
 - unreduced, 134–135, 151
- Fraenkel, Aviezri S, 515, 563, 613–614, 633
- Frame, James Sutherland, 614, 633
- Francesca, Piero della, 614, 635
- Franel, Jérôme, 549, 614
- Fraser, Alexander Yule, 2, 604
- Frazer, William Donald, 614, 634
- Fredman, Michael Lawrence, 513, 614
- free variables, 22
- Freĭman, Grigorĭi Abelevich, 608
- friendly monster, 545
- frisbees, 434–435, 437
- Frye, Roger Edward, 131
- Fundamental Theorem of Algebra, 207
- Fundamental Theorem of Arithmetic, 106–107
- Fundamental Theorem of Calculus, 48
- Fuss, Nicolaĭ Ivanovich, 361, 614
 - Fuss–Catalan numbers, 361
- Fuss, Paul Heinrich von (= Fus, Pavel Nikolaeich), 611–612

- Gale, Dorothy, 581
- games, *see* bowling, cards, cribbage, dice,
 Penney ante, sports
- Gamma function, 210–214, 609
 duplication formula for, 528
 Stirling's approximation for, 482
- gaps between primes, 150–151, 525
- Gardner, Martin, 614, 634, 636
- Garfunkel, Jack, 614, 636
- Gasper, George, Jr., 223, 614
- Gauß (= Gauss), Johann Friderich Carl (= Carl Friedrich), vii, 6, 7, 123, 205, 207, 212, 501, 510, 529, 610, 615, 633, 634
 hypergeometric series, 207
 identity for hypergeometrics, 222, 247, 539
 trick, 6, 30, 112, 313
- gcd, 103, *see* greatest common divisor
- generalization, 11, 13, 16
 downward, 2, 95, 320–321
- generalized binomial coefficients, 211, 318, 530
- generalized binomial series, 200–204, 243, 252, 363
- generalized exponential series, 200–202, 242, 364, 369
- generalized factorial function, 192, 210–211, 213–214, 316
- generalized harmonic numbers, 277, 283, 286, 370
- generalized Stirling numbers, 271–272, 311, 316, 319, 598
- generating functions, 196–204, 297–300, 320–380
 composition of, 428
 Dirichlet, 370–371, 373, 432, 451
 exponential, 364–369, 421–422
 for Bernoulli numbers, 285, 351, 365
 for convolutions, 197, 333–334, 353–364, 369, 421
 for Eulerian numbers, 351, 353
 for Fibonacci numbers, 297–300, 337–340, 351, 570
 for harmonic numbers, 351–352
 for minima, 377
 for probabilities, 394–401
 for simple sequences, 335
 for special numbers, 351–353
 for spectra, 307, 319
 for Stirling numbers, 351–352, 559
 Newtonian, 378
 of generating functions, 351, 353, 421
 super, 353, 421
 table of manipulations, 334
- Genocchi, Angelo, 615
 numbers, 551, 574
- geometric progression, 32
 floored, 114
 generalized, 205–206
 sum of, 32–33, 54
- Gessel, Ira Martin, 270, 615, 634
- Gibbs, Josiah Willard, 630
- Gilbert, William Schwenck, 444
- Ginsburg, Jekuthiel, 615
- Glaiser, James Whitbread Lee, 615, 636
 constant (≈ 1.28243), 595
- God, 1, 307, 521
- Goldbach, Christian, 611–612
 theorem, 66
- golden ratio, 299, *see* phi
- golf, 431
- Golomb, Solomon Wolf, 460, 507, 615, 633
 digit-count sum, 460–462, 490 (exercise 22), 494
 self-describing sequence, 66, 495, 630
- Good, Irving John, 615, 634
- Goodfellow, Geoffrey Scott, 628
- Gopinath, Bhaskarpillai, 501, 621
- Gordon, Peter Stuart, ix
- Gosper, Ralph William, Jr., 224, 564, 615, 634
 algorithm, 224–227
 algorithm, examples, 227–229, 245, 247–248, 253–254, 530, 534
- Gosper–Zeilberger algorithm, 229–241, 319
 examples, 254–255, 547
 summary, 233
- goto, considered harmful, 173
- Gottschalk, Walter Helbig, vii
- graffiti, vii, ix, 59, 637
- Graham, Cheryl, ix
- Graham, Ronald Lewis, iii, iv, vi, ix, 102, 506, 605, 608–609, 611, 615–616, 629, 632, 633, 635
- Grandi, Luigi Guido, 58, 616
- Granville, Andrew James, 548
- graph theory, *see* spanning trees
- graphs of functions,
 $1/x$, 276–277
 $e^{-x^2/10}$, 483
 Bernoulli polynomials, 473
 floor and ceiling, 68
 hyperbola, 440
 partial sums of a sequence, 359–360

- Graves, William Henson, 632
gravity, center of, 273–274, 309
Gray, Frank, code, 497
greatest common divisor, 92, 103–104, 107, 145
greatest integer function, *see* floor function
greatest lower bound, 65
greed, 74, 387–388; *see also* rewards
greedy algorithm, 101, 295
Green, Research Sink, 607
Greene, Daniel Hill, 616
Greitzer, Samuel Louis, 616, 633
Gross, Oliver Alfred, 616, 635
Grünbaum, Branko, 498, 616
Grundy, Patrick Michael, 627, 633
Guibas, Leonidas Ioannis (= Leo John), 590, 616, 632, 636
Guy, Richard Kenneth, 523, 525, 616
- H_n , *see* harmonic numbers
Haar, Alfréd, vii
Hacker's Dictionary, 124, 628
Haiman, Mark, 632
Håland Knutson, Inger Johanne, 616, 633
half-open interval, 73–74
Hall, Marshall, Jr., 616
Halmos, Paul Richard, v, vi, 616–617
Halphen, Georges Henri, 305, 617
halving, 79, 186–187
Hamburger, Hans Ludwig, 591, 617
Hammersley, John Michael, v, 617, 636
Hanoi, Tower of, 1–4, 26–27, 109, 146
 variations on, 17–20
Hansen, Eldon Robert, 42, 617
Hardy, Godfrey Harold, 111, 442–443, 617, 633, 636
harmonic numbers, 29, 272–282
 analogous to logarithms, 53
 asymptotics of, 276–278, 452, 480–481, 491
 complex, 311, 316
 divisibility of, 311, 314, 319
 generalized, 277, 283, 286, 370
 generating function for, 351–352
 second-order, 277, 280, 311, 550–552
 sums of, 41, 313, 316, 354–355
 sums using summation by parts, 56, 279–282, 312
 table of, 273
harmonic series, divergence of, 62, 275–276
Harry, Matthew Arnold, double sum, 249
hashing, 411–426, 430
hat-check problem, *see* football victory problem
hcf, 103, *see* greatest common divisor
Heath-Brown, David Rodney, 629
Heiberg, Johan Ludvig, 611
Heisenberg, Werner Karl, 481
Helmbold, David Paul, 632
Henrici, Peter Karl Eugen, 332, 545, 602, 617, 634, 636
Hermite, Charles, 538, 555, 617, 629, 634
herring, red, 497
Herstein, Israel Nathan, 8, 618
hexagon property, 155–156, 242, 251
highest common factor, *see* greatest common divisor
Hillman, Abraham P, 618, 634
Hoare, Sir Charles Antony Richard, 28, 73, 618, 620
Hofstadter, Douglas Richard, 633
Hoggatt, Verner Emil, Jr., 618, 623, 634
Holden, Edward Singleton, 625
Holmboe, Berndt Michael, 604
Holmes, Thomas Sherlock Scott, 162, 228–229
holomorphic functions, 196
homogeneous linear equations, 239, 543
horses, 17, 18, 468, 503
Hsu, Lee-Tsch (= Lietz = Leetch) Ching-Siur, 618, 634
Hurwitz, Adolf, 635
hyperbola, 440
hyperbolic functions, 285–286
hyperfactorial, 243, 491
hypergeometric series, 204–223
 confluent, 206, 245
 contiguous, 529
 degenerate, 209–210, 216, 222, 247
 differential equation for, 219–221
 Gaussian, 207
 partial sums of, 165–166, 223–230, 245
 transformations of, 216–223, 247, 253
hypergeometric terms, 224, 243, 245, 527, 575
 similar, 541
- i, 22
implicit recurrences, 136–139, 193–195, 284
indefinite summation, 48–49
 by parts, 54–56
 of binomial coefficients, 161, 223–224, 246, 248, 313
 of hypergeometric terms, 224–229

- independent random variables, 384, 427
 - pairwise, 437
 - products of, 386
 - sums of, 386, 396–398
- index set, 22, 30, 61
- index variable, 22, 34, 60
- induction, 3, 7, 10–11, 43
 - backwards, 18
 - basis of, 3, 320–321
 - failure of, 17, 575
 - important lesson about, 508, 549
- inductive leap, 4, 43
- infinite sums, 56–62, 64
 - doubly, 59, 98, 482–483
- information retrieval, 411–413
- INT function, 67
- insurance agents, 391
- integer part, 70
- integration, 45–46, 48
 - by parts, 54, 472
 - of generating functions, 333, 365
- interchanging the order of summation, 34–41, 105, 136, 183, 185, 546
- interpolation, 191–192
- intervals, 73–74
- invariant relation, 117
- inverse modulo m , 125, 132, 147
- inversion formulas, 193
 - for binomial coefficients, 192–196
 - for Stirling numbers, 264, 310
 - for sums over divisors, 136–139
- irrational numbers, 238
 - continued fraction representations, 306
 - rational approximations to, 122–123
 - spectra of, 77, 96, 514
 - Stern–Brocot representations, 122–123
- Iverson, Kenneth Eugene, 24, 67, 618, 633
 - convention, 24–25, 31, 34, 68, 75
- Jacobi, Carl Gustav Jacob, 64, 618
 - polynomials, 543, 605
- Janson, Carl Svante, 618
- Jarden, Dov, 556, 618
- Jeopardy, 361
- joint distribution, 384
- Jonassen, Arne Tormod, 618
- Jones, Bush, 618
- Josephus, Flavius, 8, 12, 19–20, 618
 - numbers, 81, 97, 100
 - problem, 8–17, 79–81, 95, 100, 144
 - recurrence, generalized, 13–16, 79–81, 498
 - subset, 20
- Jouailllec, Louis Maurice, 632
- Jungen, Reinwald, 618, 635
- K, *see* continuants
- Kaplansky, Irving, 8, 568, 618
- Karamata, Jovan, 257, 618
- Karlin, Anna Rochelle, 632
- Kaucký, Josef, 619, 635
- Kauers, Manuel, 564
- Keiper, Jerry Bruce, 619
- Kellogg, Oliver Dimon, 609
- Kent, Clark (= Kal-El), 372
- Kepler, Johannes, 292, 619
- kernel functions, 370
- Ketcham, Henry King, 148
- kilometers, 301, 310, 550
- Kilroy, James Joseph, vii
- Kipling, Joseph Rudyard, 260
- Kissinger, Henry Alfred, 379
- Klamkin, Murray Seymour, 619, 633, 635
- Klarner, David Anthony, 632
- knockout tournament, 432–433
- Knoebel, Robert Arthur, 619
- Knopp, Konrad, 619, 636
- Knuth, Donald Ervin, iii–ix, 102, 267, 411, 506, 553, 616, 618–620, 632, 633, 636, 657
 - numbers, 78, 97, 100
- Knuth, John Martin, 636
- Knuth, Nancy Jill Carter, ix
- Kramp, Christian, 111, 620
- Kronecker, Leopold, 521
 - delta notation, 24
- Kruk, John Martin, 519
- Kummer, Ernst Eduard, 206, 529, 621, 634
 - formula for hypergeometrics, 213, 217, 535
- Kurshan, Robert Paul, 501, 621
- L_n , *see* Lucas numbers
- Lagny, Thomas Fantet de, 304, 621
- Lagrange (= de la Grange), Joseph Louis, comte, 470, 621, 635
 - identity, 64
- Lah, Ivo, 621, 635
- Lambert, Johann Heinrich, 201, 363, 613, 621
- Landau, Edmund Georg Hermann, 443, 448, 622, 634, 636
- Laplace, Pierre Simon, marquis de, 466, 606, 622
- last but not least, 132, 469

- Law of Large Numbers, 391
 lcm, 103, *see* least common multiple
 leading coefficient, 235
 least common multiple, 103, 107, 145
 of $\{1, \dots, n\}$, 251, 319, 500
 least integer function, *see* ceiling function
 least upper bound, 57, 61
 LeChiffre, Mark Well, 148
 left-to-right maxima, 316
 Legendre, Adrien Marie, 622, 633
 polynomials, 543, 573, 575
 Lehmer, Derrick Henry, 526, 622, 633, 635
 Leibniz, Gottfried Wilhelm, Freiherr von, vii,
 168, 616, 622
 Lekkerkerker, Cornelis Gerrit, 622
 Lengyel, Tamás Lóránt, 622, 635
 levels of problems, viii, 72–73, 95, 511
 Levine, Eugene, 611, 635
 lexicographic order, 441
 lg: binary logarithm, 70, 449
 L'Hospital, Guillaume François Antoine de,
 marquis de Sainte Mesme, rule, 340,
 396, 542
 Lǐ Shànlán (= Rénshū = Qiūrèn), 269, 622
 Liang, Franklin Mark, 632
 Lieb, Elliott Hershel, 622, 636
 lies, and statistics, 195
 Lincoln, Abraham, 401
 linear difference operators, 240
 lines in the plane, 4–8, 17, 19
 Liouville, Joseph, 136–137, 622
 little oh notation, 448
 considered harmful, 448–449
 Littlewood, John Edensor, 239
 ln: natural logarithm, 276, 449
 discrete analog of, 53–54
 sum of, 481–482
 log: common logarithm, 449
 Logan, Benjamin Franklin (= Tex), Jr., 287,
 622–623, 634–635
 logarithmico-exponential functions, 442–443
 logarithms, 449
 binary, 70
 discrete analog of, 53–54
 in O-notation, 449
 natural, 276
 Long, Calvin Thomas, 623, 634
 lottery, 387–388, 436–437
 Lóu, Shìtuó, 623
 lower index of binomial coefficient, 154
 complex valued, 211
 lower parameters of hypergeometric series, 205
 Loyd, Samuel, 560, 623
 Lucas, François Édouard Anatole, 1, 292, 623,
 633–635
 numbers, 312, 316, 556
 Łuczak, Tomasz Jan, 618
 Lyness, Robert Cranston, 501, 623
 Maclaurin (= Mac Laurin), Colin, 469, 623
 MacMahon, Maj. Percy Alexander, 140, 623
 magic tricks, 293
 Mallows, Colin Lingwood, 506
 Markov, Andreĭ Andreevich (the elder), pro-
 cesses, 405
 Martian DNA, 377
 Martzloff, Jean-Claude, 623
 mathematical induction, 3, 7, 10–11, 43
 backwards, 18
 basis of, 3, 320–321
 failure of, 17, 575
 important lesson about, 508, 549
 Mathews, Edwin Lee (= 41), 8, 21, 94, 105,
 106, 343
 Matiĭasevich (= Matijasevich), Īuriĭ (= Yuri)
 Vladimirovich, 294, 623, 635
 Mauldin, Richard Daniel, 611
 Maxfield, Margaret Waugh, 630, 635
 Mayr, Ernst, ix, 632, 633
 McEliece, Robert James, 71
 McGrath, James Patrick, 632
 McKellar, Archie Charles, 614, 634
 mean (average) of a probability distribution,
 384–399
 median, 384, 385, 437
 mediant, 116
 Melzak, Zdzislaw Alexander, vi, 623
 Mendelsohn, Nathan Saul, 623, 634
 Merchant, Arif Abdulhussein, 632
 merging, 79, 175
 Mersenne, Marin, 109–110, 131, 613, 624
 numbers, 109–110, 151, 292
 primes, 109–110, 127, 522–523
 Mertens, Franz Carl Joseph, 23, 139, 624
 miles, 301, 310, 550
 Mills, Stella, 624
 Mills, William Harold, 624, 634
 minimum, 65, 249, 377
 Minkowski, Hermann, 122

- Mirsky, Leon, 635
 mixture of probability distributions, 428
 mnemonics, 74, 164
 Möbius, August Ferdinand, 136, 138, 624
 function, 136–139, 145, 149, 370–371, 462–463
 mod: binary operation, 81–85
 mod: congruence relation, 123–126
 mod 0, 82–83, 515
 mode, 384, 385, 437
 modular arithmetic, 123–129
 modulus, 82
 Moessner, Alfred, 624, 636
 Moivre, Abraham de, 297, 481, 609
 moments, 398–399
 Montgomery, Hugh Lowell, 463, 624
 Montgomery, Peter Lawrence, 624, 634
 Moriarty, James, 162
 Morse, Samuel Finley Breese, code, 302–303, 324, 551
 Moser, Leo, 624, 633
 Motzkin, Theodor Samuel, 556, 564, 618, 624
 mountain ranges, 359, 565
 mu function, *see* Möbius function
 multinomial coefficients, 168, 171–172, 569
 recurrence for, 252
 multinomial theorem, 149, 168
 multiple of a number, 102
 multiple sums, 34–41, 61; *see also* double sums
 multiple-precision numbers, 127
 multiplicative functions, 134–136, 144, 371
 multisets, 77, 270
 mumble function, 83, 84, 88, 507, 513
 Murdock, Phoebe James, viii
 Murphy's Law, 74
 Myers, Basil Roland, 624, 635
 name and conquer, 2, 32, 88, 139
 National Science Foundation, ix
 natural logarithm, 53–54, 276, 449, 481–482
 Naval Research, ix
 navel research, 299
 nearest integer, 95
 rounding to, 195, 300, 344, 491
 unbiased, 507
 necessary and sufficient conditions, 72
 necklaces, 139–141, 259
 negating the upper index, 164–165
 negative binomial distribution, 402–403, 428
 negative factorial powers, 52, 63, 188
 Newman, James Roy, 631
 Newman, Morris, 635
 Newton, Sir Isaac, 189, 277, 624
 series, 189–192
 Newtonian generating function, 378
 Niven, Ivan Morton, 332, 624, 633
 nonprime numbers, 105, 518
 nontransitive paradox, 410
 normal distribution, 438
 notation, x -xi, 2, 637
 extension of, 49, 52, 154, 210–211, 266, 271, 311, 319
 ghastly, 67, 175
 need for new, 83, 115, 267
 nu function: sum of digits,
 binary (radix 2), 12, 114, 250, 525, 557
 other radices, 146, 525, 552
 null case, for spanning trees, 349, 565
 for Stirling numbers, 258
 for tilings, 320–321
 for Tower of Hanoi, 2
 number system, 107, 119
 combinatorial, 245
 Fibonacci, 296–297, 301, 307, 310, 318
 prime-exponent, 107, 116
 radix, *see* radix notation
 residue, 126–129, 144
 Stern–Brocot, *see* Stern–Brocot number system
 number theory, 102–152
 o, considered harmful, 448–449
 O-notation, 76, 443–449
 abuse of, 447–448, 489
 one-way equalities with, 446–447, 489–490
 obvious, clarified, 417, 526
 odds, 410
 Odlyzko, Andrew Michael, 81, 564, 590, 616, 624, 636
 Office of Naval Research, ix
 one-way equalities, 446–447, 489–490
 open interval, 73–74, 96
 operators, 47
 anti-derivative (\int), 48
 anti-difference (\sum), 48
 derivative (D), 47, 310
 difference (Δ), 47
 equations of, 188, 191, 241, 310, 471
 shift (E, K, N), 55, 240
 theta (θ), 219, 310
 optical illusions, 292, 293, 560

- organ-pipe order, 524
- Oz, Wizard of, 581
- Pacioli, Luca, 614
- Palais, Richard Sheldon, viii
- paradoxes,
 - chessboard, 293, 317
 - coin flipping, 408–410
 - pair of boxes, 531, 535, 539
- paradoxical sums, 57
- parallel summation, 159, 174, 208–210
- parentheses, 357–359
- parenthesis conventions, xi
- partial fraction expansions, 298–299, 338–341
 - for easy summation and differentiation, 64, 376, 476, 504, 586
 - not always easiest, 374
 - of $1/x^{\binom{x+n}{n}}$, 189
 - of $1/(z^n - 1)$, 558
 - powers of, 246, 376
- partial quotients, 306
 - and discrepancies, 319, 598–599, 602
 - large, 553, 563, 564, 602
- partial sums, *see* indefinite summation
 - required to be positive, 359–362
- partition into nearly equal parts, 83–85
- partitions, of the integers, 77–78, 96, 99, 101
 - of a number, 330, 377
 - of a set, 258–259, 373
- Pascal, Blaise, 155, 156, 624–625, 633
- Pascal's triangle, 155
 - extended upward, 164
 - hexagon property, 155–156, 242, 251
 - row lcms, 251
 - row products, 243
 - row sums, 163, 165–166
 - variant of, 250
- Patashnik, Amy Markowitz, ix
- Patashnik, Oren, iii, iv, vi, ix, 102, 506, 616, 632
- Patil, Ganapati Parashuram, 625, 636
- Paule, Peter, 537, 546
- Peirce, Charles Santiago Sanders, 151, 525, 625, 634
- Penney, Walter Francis, 408, 625
- Penney ante, 408–410, 430, 437, 438
- pentagon, 314 (exercise 46), 430, 434
- pentagonal numbers, 380
- Percus, Jerome Kenneth, 625, 636
- perfect powers, 66
- periodic recurrences, 20, 179, 498
- permutations, 111–112
 - ascents in, 267–268, 270
 - cycles in, 259–262
 - excedances in, 316
 - fixed points in, 193–196, 393–394, 400–401, 428
 - left-to-right maxima in, 316
 - random, 393–394, 400–401, 428
 - up-down, 377
 - without fixed points, *see* derangements
- personal computer, 109
- perspiration, 234–235
- perturbation method, 32–33, 43–44, 64, 179, 284–285
- Petkovšek, Marko, 229, 575, 625, 634
- Pfaff, Johann Friedrich, 207, 214, 217, 529, 625, 634
 - reflection law, 217, 244, 247, 539
- pgf: probability generating function, 394
- phages, 434, 438
- phi (≈ 1.61803), 299–301
 - as canonical constant, 70
 - continued fraction for, 310
 - in fifth roots of unity, 553
 - in solutions to recurrences, 97, 99, 299–301
 - Stern–Brocot representation of, 550
- phi function, 133–135
 - dgf for, 371
 - divisibility by, 151
- Phi function: sum of φ , 138–139, 462–463
- Phidias, 299
- philosophy, vii, 11, 16, 46, 71, 72, 75, 91, 170, 181, 194, 331, 467, 503, 508, 603
- phyllotaxis, 291
- pi (≈ 3.14159), 26, 286
 - as canonical constant, 70, 416, 423
 - large partial quotients of, 564
 - Stern–Brocot representation of, 146
- pi function, 110–111, 452, 593
 - preposterous expressions for, 516
- Pig, Porky, 496
- pigeonhole principle, 130
- Pincherle, Salvatore, 617
- Pisano, Leonardo filio Bonacii, 613, *see* Fibonacci
- Pittel, Boris Gershon, 576, 618
- pizza, 4, 423
- planes, cutting, 19
- Plouffe, Simon, 628

- pneumathics, 164
- Pochhammer, Leo, 48, 625
symbol, 48
- pocket calculators, 67, 77, 459
failure of, 344
- Poincaré, Jules Henri, 625, 636
- Poisson, Siméon Denis, 471, 625
distribution, 428–429, 579
summation formula, 602
- Pollak, Henry Otto, 616, 633
- Pólya, George (= György), vi, 16, 327, 508,
625–626, 633, 635, 636
- polygons, dissection of, 379
triangulation of, 374
Venn diagrams with, 20
- polynomial argument, 158, 163
for rational functions, 527
opposite of, 210
- polynomially recursive sequence, 374
- polynomials, 189
Bernoulli, 367–368, 470–475
continuant, 301–309
convolution, 373
cyclotomic, 149
degree of, 158, 226
divisibility of, 225
Euler, 574
Jacobi, 543, 605
Legendre, 543, 573, 575
Newton series for, 189–191
reflected, 339
Stirling, 271–272, 290, 311, 317, 352
- Poonen, Bjorn, 501, 633
- Poorten, Alfred Jacobus van der, 630
- Porter, Thomas K, 632
- Portland cement, *see* concrete (in another book)
- power series, 196, *see* generating functions
formal, 206, 331, 348, 532
- Pr, 381–382
- Pratt, Vaughan Ronald, 632
- preferential arrangements, 378 (exercise 44)
- primality testing, 110, 148
impractical method, 133
- prime algebraic integers, 106, 147
- prime numbers, 105–111
gaps between, 150–151, 525
largest known, 109–110
Mersenne, 109–110, 127, 522–523
size of n th, 110–111, 456–457
sum of reciprocals, 22–25
- prime to, 115
- prime-exponent representation, 107, 116
- Princeton University, ix, 427
- probabilistic analysis of an algorithm, 413–426
- probability, 195, 381–438
conditional, 416–419, 424–425
discrete, 381–438
generating functions, 394–401
spaces, 381
- probability distributions, 381
binomial, 401–402, 415, 428, 432
composition or mixture of, 428
joint, 384
negative binomial, 402–403, 428
normal, 438
Poisson, 428–429, 579
uniform, 395–396, 418–421
- problems, levels of, viii, 72–73, 95, 511
- Proding, Helmut, 564
- product notation, 64, 106
- product of consecutive odd numbers, 186, 270
- progression, *see* arithmetic progression, geomet-
ric progression
- proof, 4, 7
- proper terms, 239–241, 255–256
- properties, 23, 34, 72–73
- prove or disprove, 71–72
- psi function, 551
- pulling out the large part, 453, 458
- puns, ix, 220
- Pythagoras of Samos, theorem, 510
- quadratic domain, 147
- quicksort, 28–29, 54
- quotation marks, xi
- quotient, 81
- rabbits, 310
- radix notation, 11–13, 15–16, 109, 195, 526
length of, 70, 460
related to prime factors, 113–114, 146–148,
245
- Rado, Richard, 625, 635
- Rahman, Mizan, 223, 614
- Rainville, Earl David, 529, 626
- Ramanujan Iyengar, Srinivasa, 330
- Ramaré, Olivier, 548
- Ramshaw, Lyle Harold, 73, 632, 634, 636
- random constant, 399

- random variables, 383–386; *see also* independent random variables
- Raney, George Neal, 359, 362, 626, 635
 - lemma, 359–360
 - lemma, generalized, 362, 372
 - sequences, 360–361
- Rao, Dekkata Rameswar, 626, 633
- rational functions, 207–208, 224–226, 338, 527
- rational generating functions, 338–346
 - expansion theorems for, 340–341
- Rayleigh, John William Strutt, 3rd Baron, 77, 626
- Read, Ronald Cedric, 625
- real part, 64, 212, 451
- reciprocity law, 94
- Recorde, Robert, 446, 626
- recurrences, 1–20
 - and sums, 25–29
 - doubly exponential, 97, 100, 101, 109
 - floor/ceiling, 78–81
 - implicit, 136–139, 193–195, 284
 - periodic, 20, 179, 498
 - solving, 337–350
 - unfolding, 6, 100, 159–160, 312
 - unfolding asymptotically, 456
- referee, 175
- reference books, 42, 223, 616, 619
- reflected light rays, 291–292
- reflected polynomials, 339
- reflection law for hypergeometrics, 217, 247, 539
- regions, 4–8, 17, 19
- regular expressions, 278
- Reich, Simeon, 626, 636
- relative error, 452, 455
- relatively prime integers, 108, 115–123
- remainder after division, 81–82
- remainder in Euler's summation formula, 471, 474–475, 479–480
- Rémy, Jean-Luc, 603
- Renz, Peter Lewis, viii
- repertoire method, 14–15, 19, 250
 - for Fibonacci-like recurrences, 312, 314, 372
 - for sums, 26, 44–45, 63
- replicative function, 100
- repunit primes, 516
- residue calculus, 495
- residue number system, 126–129, 144
- retrieving information, 411–413
- rewards, monetary, ix, 256, 497, 525, 575
- Rham, Georges de, 626, 635
- Ribenboim, Paulo, 555, 626, 634
- Rice, Stephan Oswald, 626
- Rice University, ix
- Riemann, Georg Friedrich Bernhard, 205, 626, 633
 - hypothesis, 526
- Riemann's zeta function, 65, 595
 - as generalized harmonic number, 277–278, 286
 - as infinite product, 371
 - as power series, 601
 - dgf's involving, 370–371, 373, 463, 566, 569
 - evaluated at integers, 238, 286, 571, 595, 597
- rising factorial powers, 48
 - binomial theorem for, 245
 - complex, 211
 - negative, 63
 - related to falling powers, 63, 312
 - related to ordinary powers, 263, 598
- Roberts, Samuel, 626, 633
- rocky road, 36, 37
- Rødseth, Øystein Johan, 627, 634
- Rolletschek, Heinrich Franz, 514
- roots of unity, 149, 204, 375, 574, 598
 - fifth, 553
 - modulo m , 128–129
- Roscoe, Andrew William, 620
- Rosser, John Barkley, 111, 627
- Rota, Gian-Carlo, 516, 627
- roulette wheel, 74–76, 453
- rounding to nearest integer, 95, 195, 300, 344, 491
 - unbiased, 507
- Roy, Ranjan, 627, 634
- rubber band, 274–275, 278, 312, 493
- ruler function, 113, 146, 148
- running time, 413, 425–426
 - O-notation for, abused, 447–448
- Ruzsa, Imre Zoltán, 611
- Saalschütz, Louis, 214, 627
 - identity, 214–215, 234–235, 529, 531
- Saltykov, Al'bert Ivanovich, 463, 627
- sample mean and variance, 391–393, 427
- sample third cumulant, 429
- samplesort, 354
- sandwiching, 157, 165
- Sárközy, András, 548, 627
- Sawyer, Walter Warwick, 207, 627
- Schäffer, Alejandro Alberto, 632

- Schinzel, Andrzej, 525
 Schlömilch, Oscar Xaver, 627
 Schmidt, Asmus Lorenzen, 634
 Schoenfeld, Lowell, 111, 627
 Schönheim, Johanan, 608
 Schröder, Ernst, 627, 635
 Schrödinger, Erwin, 430
 Schröter, Heinrich Eduard, 627, 635
 Schützenberger, Marcel Paul, 636
 science and art, 234
 Scorer, Richard Segar, 627, 633
 searching a table, 411–413
 Seaver, George Thomas (= 41), 8, 21, 94, 105, 106, 343
 secant numbers, 317, 559, 570, 620
 second-order Eulerian numbers, 270–271
 second-order Fibonacci numbers, 375
 second-order harmonic numbers, 277, 280, 311, 550–552
 Sedgewick, Robert, 632
 Sedláček, Jiří, 627, 635
 Seidel, Philipp Ludwig von, 605
 self-certifying algorithms, 104
 self-describing sequence, 66, 495
 self reference, 59, 95, 531–540, 616, 653
 set inclusion in O-notation, 446–447, 490
 Shallit, Jeffrey Outlaw, 627, 635
 Sharkansky, Stefan Michael, 632
 Sharp, Robert Thomas, 273, 627
 sherry, 433
 shift operator, 55, 240
 binomial theorems for, 188, 191
 Shiloach, Joseph (= Yossi), 632
 Shor, Peter Williston, 633
 Sicherman, George Leprechaun, 636
 sideways addition, 12, 114, 146, 250, 552
 Sierpiński, Waclaw Franciszek, 87, 627–628, 634
 sieve of Eratosthenes, 111
 Sigma-notation, 22–25
 ambiguity of, 245
 signum function, 502
 Silverman, David L, 628, 635
 similar hypergeometric terms, 541
 skepticism, 71
 Skiena, Steven Sol, 548
 Sloane, Neil James Alexander, 42, 341, 464, 604, 628, 633
 Slowinski, David Allen, 109
 small cases, 2, 5, 9, 155, 320–321; *see also*
 empty case
- Smith, Cedric Austen Bardell, 627, 633
 Snowwalker, Luke, 435
 Solov'ev, Aleksandr Danilovitch, 408, 628
 solution, 3, 337
 sorting,
 asymptotic efficiency of, 447–449
 bubblesort, 448
 merge sort, 79, 175
 possible outcomes, 378
 quicksort, 28–29, 54
 samplesort, 354
 Soundararajan, Kannan, 525, 605.
 spanning trees,
 of complete graphs, 368–369
 of fans, 348–350, 356
 of wheels, 374
 Spec, *see* spectra
 special numbers, 257–319
 spectra, 77–78, 96, 97, 99, 101
 generating functions for, 307, 319
 spinning coins, 401
 spiral function, 99
 Spohn, William Gideon, Jr., 628
 sports, *see* baseball, football, frisbees, golf, tennis
 square pyramidal numbers, 42
 square root,
 of 1 (mod m), 128–129
 of 2, 100
 of 3, 378
 of -1 , 22
 squarefree, 145, 151, 373, 525, 548
 squares, sum of consecutive, 41–46, 51, 180, 245, 269, 284, 288, 367, 444, 470
 stack size, 360–361
 stacking bricks, 313, 374
 stacking cards, 273–274, 278, 309
 Stallman, Richard Matthew, 628
 standard deviation, 388, 390–394
 Stanford University, v, vii, ix, 427, 458, 632, 634, 657
 Stanley, Richard Peter, 270, 534, 615, 628, 635, 636
 Staudt, Karl Georg Christian von, 628, 635
 Staver, Tor Bøhm, 628, 634
 Steele, Guy Lewis, Jr., 628
 Stegun, Irene Anne, 42, 604
 Stein, Sherman Kopald, 633
 Steiner, Jacob, 5, 628, 633
 Steinhaus, Hugo Dyonizy, 636

- Stengel, Charles Dillon (= Casey), 42
- step functions, 87
- Stern, Moritz Abraham, 116, 629
- Stern–Brocot number system, 119–123
 related to continued fractions, 306
 representation of $\sqrt{3}$, 572
 representation of γ , 306
 representation of π , 146
 representation of ϕ , 550
 representation of e , 122, 150
 simplest rational approximations from, 122–123, 146, 519
- Stern–Brocot tree, 116–123, 148, 525
 largest denominators in, 319
 related to continued fractions, 305–306
- Stern–Brocot wreath, 515
- Stewart, Bonnie Madison, 614, 633
- Stickelberger, Ludwig, 629, 633
- Stieltjes, Thomas Jan, 617, 629, 633
 constants, 595, 601
- Stirling, James, 192, 195, 210, 257, 258, 297, 481, 629
 approximation, 112, 452, 481–482, 491, 496
 approximation, perturbed, 454–455
 constant, 481, 485–489
 polynomials, 271–272, 290, 311, 317, 352
 triangles, 258, 259, 267
- Stirling numbers, 257–267
 as sums of products, 570
 asymptotics of, 495, 602
 combinatorial interpretations, 258–262
 convolution formulas, 272, 290
 duality of, 267
 generalized, 271–272, 311, 316, 319, 598
 generating functions for, 351–352, 559
 identities for, 264–265, 269, 272, 290, 311, 317, 378
 inversion formulas for, 310
 of the first kind, 259
 of the second kind, 258
 related to Bernoulli numbers, 289–290, 317 (exercise 76)
 table of, 258, 259, 267
- Stone, Marshall Harvey, vi
- Straus, Ernst Gabor, 564, 611, 624
- Strehl, Karl Ernst Volker, 549, 629, 634
- Stueben, Michael A., 445
- subfactorial, 194–196, 250
- summand, 22
- summation, 21–66
 asymptotic, 87–89, 466–496
 by parts, 54–56, 63, 279
 changing the index of, 30–31, 39
 definite, 49–50, 229–241
 difficulty measure for, 181
 factors, 27–29, 64, 236, 248, 275, 543
 in hypergeometric terms, 224–229
 indefinite, *see* indefinite summation
 infinite, 56–62, 64
 interchanging the order of, 34–41, 105, 136, 183, 185, 546
 mechanical, 229–241
 on the upper index, 160–161, 175–176
 over divisors, 104–105, 135–137, 141, 370
 over triangular arrays, 36–41
 parallel, 159, 174, 208–210
- sums, 21–66; *see also* summation
 absolutely convergent, 60–62, 64
 and recurrences, 25–29
 approximation of, by integrals, 45, 276–277, 469–475
 divergent, *see* divergent sums
 double, *see* double sums
 doubly infinite, 59, 98, 482–483
 empty, 24, 48
 floor/ceiling, 86–94
 formal, 321; *see also* formal power series
 hypergeometric, *see* hypergeometric series
 infinite, 56–62, 64
 multiple, 34–41, 61; *see also* double sums
 notations for, 21–25
 of consecutive cubes, 51, 63, 283, 289, 367
 of consecutive integers, 6, 44, 65
 of consecutive m th powers, 42, 283–285, 288–290, 366–368
 of consecutive squares, 41–46, 51, 180, 245, 269, 284, 288, 367, 444, 470
 of harmonic numbers, 41, 56, 279–282, 312–313, 316, 354–355
 paradoxical, 57
 tails of, 466–469, 488–489, 492
- Sun Tsü (= Sünzǐ, Master Sun), 126
- sunflower, 291
- super generating functions, 353, 421
- superfactorials, 149, 243
- Swanson, Ellen Esther, viii
- Sweeney, Dura Warren, 629
- Swinden, Benjamin Alfred, 633
- Sylvester, James Joseph, 133, 629, 633

- symmetry identities,
 - for binomial coefficients, 156–157, 183
 - for continuants, 303
 - for Eulerian numbers, 268
- Szegedy, Máriaó, 525, 608, 629
- Szegő, Gábor, 626, 636
- T_n , *see* tangent numbers
- tail exchange, 466–469, 486–489
- tail inequalities, 428, 430
- tail of a sum, 466–469, 488–489, 492
- tale of a sum, *see* squares
- Tancke, Joachim, 619
- tangent function, 287, 317
- tangent numbers, 287, 312, 317, 570, 620
- Tanny, Stephen Michael, 629, 635
- Tartaglia, Nicolò, triangle, 155
- Taylor, Brook, series, 163, 191, 287, 396, 470–471
- telescoping, 50, 232, 236, 255
- tennis, 432–433
- term, 21
 - hypergeometric, 224, 243, 245, 527, 575
- term ratio, 207–209, 211–212, 224–225
- \TeX , 219, 432, 657
- Thackeray, Henry St. John, 618
- Theisinger, Ludwig, 629, 634
- theory of numbers, 102–152
- theory of probability, 381–438
- theta functions, 483, 524
- theta operator, 219–221, 347
 - converting between D and ϑ , 310
- Thiele, Thorvald Nicolai, 397, 398, 629
- thinking, 503
 - big, 2, 441, 458, 483, 486
 - not at all, 56, 230, 503
 - small, *see* downward generalization, small cases
- three-dots (\dots) notation, 21
 - advantage of, 21, 25, 50
 - disadvantage of, 25
 - elimination of, 108
- tilings, *see* domino tilings
- Titchmarsh, Edward Charles, 629, 636
- Todd, Horace, 501
- Toledo, Ohio, 73
- Tong, Christopher Hing, 632
- Toscano, Letterio, 621
- totient function, 133–135
 - dgf for, 371
 - divisibility by, 151
 - summation of, 137–144, 150, 462–463
- Toto, 581
- tournament, 432–433
- Tower of Brahma, 1, 4, 278
- Tower of Hanoi, 1–4, 26–27, 109, 146
 - variations on, 17–20
- Trabb Pardo, Luis Isidoro, 632
- transitive law, 124
 - failure of, 410
- traps, 154, 157, 183, 222, 542
- trees,
 - 2-3 trees, 636
 - binary, 117
 - of bees, 291
 - spanning, 348–350, 356, 368–369, 374
 - Stern–Brocot, *see* Stern–Brocot tree
- triangular array, summation over, 36–41
- triangular numbers, 6, 155, 195–196, 260, 380
- triangulation, 374
- Tricomi, Francesco Giacomo Filippo, 629, 636
- tridiagonal matrix, 319
- trigonometric functions,
 - related to Bernoulli numbers, 286–287, 317
 - related to probabilities, 435, 437
 - related to tilings, 379
- trinomial coefficients, 168, 171, 255, 571
 - middle, 490
- trinomial theorem, 168
- triphages, 434
- trivial, clarified, 105, 129, 417–418, 618
- Turán, Paul, 636
- typefaces, viii–ix, 657
- Uchimura, Keisuke, 605, 635
- umop-apısdn function, 193
- unbiased estimate, 392, 429
- unbiased rounding, 507
- uncertainty principle, 481
- undetermined coefficients, 529
- unexpected sum, 167, 215–216, 236, 247
- unfolding a recurrence, 6, 100, 159–160, 312
 - asymptotically, 456
- Ungar, Peter, 629
- uniform distribution, 395–396, 418–421
- uniformity, deviation from, 152; *see also* discrepancy
- unique factorization, 106–107, 147
- unit, 147
- unit fractions, 95, 101, 150
- unwinding a recurrence, *see* unfolding a recurrence

- up-down permutations, 377
- upper index of binomial coefficient, 154
- upper negation, 164–165
- upper parameters of hypergeometric series, 205
- upper summation, 160–161, 176
- useless identity, 223, 254
- Uspensky, James Victor, 615, 630, 633
- V: variance, 387–398, 419–425
- van der Poorten, Alfred Jacobus, 630
- Vandermonde, Alexandre Théophile, 169, 630, 634
- Vandermonde's convolution, 169–170, 610, 627
 - as a hypergeometric series, 211–213
 - combinatorial interpretation, 169–170
 - derived mechanically, 234
 - derived from generating functions, 198
 - generalized, 201–202, 218–219, 248
 - with half-integers, 187
- vanilla, 36
- Vardi, Ilan, 525, 548, 603, 620, 630, 633, 636
- variance of a probability distribution, 387–398, 419–425
 - infinite, 428, 587
- Veech, William Austin, 514
- Venn, John, 498, 630, 633
 - diagram, 17, 20
- venture capitalists, 493–494
- violin string, 29
- vocabulary, 75
- Voltaire, de (= Arouet, François Marie), 450
- von Seidel, Philipp Ludwig, 605
- von Staudt, Karl Georg Christian, 628, 635
- Vyssotsky, Victor Alexander, 548
- Wall, Charles Robert, 607, 635
- Wallis, John, 630, 635
- Wapner, Joseph Albert, 43
- war, 8, 16, 85, 434
- Waring, Edward, 630, 635
- Wasteels, Joseph, 630, 635
- Waterhouse, William Charles, 630, 635
- Watson, John Hamish, 229, 405
- Waugh, Frederick Vail, 630, 635
- Weaver, Warren, 630
- Weber, Heinrich, 630
- Weisner, Louis, 516, 630
- Wermuth, Edgar Martin Emil, 603, 630
- Weyl, Claus Hugo Hermann, 87, 630
- Wham-O, 435, 443
- wheel, 74, 374
 - big, 75
 - of Fortune, 453
- Whidden, Samuel Blackwell, viii
- Whipple, Francis John Welsh, 630, 634
 - identity, 253
- Whitehead, Alfred North, 91, 503, 603, 631
- Wiles, Andrew John, 131
- Wilf, Herbert Saul, 81, 240, 241, 514, 549, 575, 620, 624, 631, 634
- Williams, Hugh Cowie, 631, 633
- Wilquin, Denys, 634
- Wilson, George and Martha, 148
- Wilson, Sir John, theorem, 132–133, 148, 516, 609
- wine, 433
- Witty, Carl Roger, 509
- Wolstenholme, Joseph, 631, 635
 - theorem, 554
- Wood, Derick, 631, 633
- Woods, Donald Roy, 628
- Woolf, William Blauvelt, viii
- worm,
 - and apple, 430
 - on rubber band, 274–275, 278, 312, 493
- Worpitzky, Julius Daniel Theodor, 631
 - identity, 269
- wraparound, 250 (exercise 75), 315
- wreath, 515
- Wrench, John William, Jr., 600, 606, 636
- Wright, Sir Edward Maitland, 111, 617, 631, 633
- Wythoff (= Wijthoff), Willem Abraham, 614
- Yao, Andrew Chi-Chih, ix, 632
- Yao, Frances Foong Chu, ix, 632
- Yáo, Qí, 623
- Youngman, Henry (= Henny), 175
- zag, *see* zig
- Zagier, Don Bernard, 238
- Zapf, Hermann, viii, 620, 657
- Zave, Derek Alan, 631, 635
- Zeckendorf, Edouard, 631
 - theorem, 295–296, 563
- Zeilberger, Doron, ix, 229–231, 238, 240, 241, 631, 634
- zero, not considered harmful, 24–25, 159
 - strongly, 24–25
- zeta function, 65, 595
 - and the Riemann hypothesis, 526
 - as generalized harmonic number, 277–278, 286
 - as infinite product, 371
 - as power series, 601
 - dgf's involving, 370–371, 373, 463, 566, 569
 - evaluated at integers, 238, 286, 571, 595, 597
- Zhu Shijie, *see* Chu Shih-Chieh
- zig, 7–8, 19
- zig-zag, 19
- Zipf, George Kingsley, law, 419

List of Tables

Sums and differences	55
Pascal's triangle	155
Pascal's triangle extended upward	164
Sums of products of binomial coefficients	169
The top ten binomial coefficient identities	174
General convolution identities	202
Stirling's triangle for subsets	258
Stirling's triangle for cycles	259
Basic Stirling number identities	264
Additional Stirling number identities	265
Stirling's triangles in tandem	267
Euler's triangle	268
Second-order Eulerian triangle	270
Stirling convolution formulas	272
Generating function manipulations	334
Simple sequences and their generating functions	335
Generating functions for special numbers	351
Asymptotic approximations	452

THIS BOOK was composed at Stanford University using the \TeX system for technical text developed by D. E. Knuth. The mathematics is set in a new typeface called AMS Euler (Version 2.1), designed by Hermann Zapf for the American Mathematical Society. The text is set in a new typeface called Concrete Roman and Italic, a special version of Knuth's Computer Modern family with weights designed to blend with AMS Euler. The paper is 50-lb.-basis Bright White Finch Opaque, which has a neutral pH and a life expectancy of several hundred years. The offset printing and notch binding were done by the Courier Corporation of Westford, Massachusetts.

This book introduces the mathematics that supports advanced computer programming and the analysis of algorithms. The primary aim of its well-known authors is to provide a solid and relevant base of mathematical skills—the skills needed to solve complex problems, to evaluate horrendous sums, and to discover subtle patterns in data. It is an indispensable text and reference not only for computer scientists—the authors themselves rely heavily on it!—but for serious users of mathematics in virtually every discipline.

Concrete mathematics is a blending of CONTINUOUS and DISCRETE mathematics. “More concretely,” the authors explain, “it is the controlled manipulation of mathematical formulas, using a collection of techniques for solving problems.” The subject matter is primarily an expansion of the Mathematical Preliminaries section in Knuth’s classic *Art of Computer Programming*, but the style of presentation is more leisurely, and individual topics are covered more deeply. Several new topics have been added, and the most significant ideas have been traced to their historical roots. The book includes more than 500 exercises, divided into six categories. Complete answers are provided for all exercises, except research problems, making the book particularly valuable for self-study.

Major topics include:

Sums • Recurrences • Integer functions • Elementary number theory • Binomial coefficients • Generating functions • Discrete probability • Asymptotic methods

This second edition includes important new material about mechanical summation. In response to the widespread use of the first edition as a reference book, the bibliography and index have also been expanded, and additional nontrivial improvements can be found on almost every page.

Readers will appreciate the informal style of *Concrete Mathematics*. Particularly enjoyable are the marginal graffiti contributed by students who have taken courses based on this material. Graham, Knuth, and Patashnik want to convey not only the importance of the techniques presented, but some of the fun in learning and using them.

About the authors:

RONALD L. GRAHAM is Chief Scientist at AT&T Labs Research. He is also University Professor of Mathematical Sciences at Rutgers University, and a former President of the American Mathematical Society. Dr. Graham is the author of six other mathematics books.

DONALD E. KNUTH is Professor Emeritus of The Art of Computer Programming at Stanford University. His prolific writings include three volumes on the *Art of Computer Programming*, and five books related to his T_EX and META-FONT typesetting systems.

OREN PATASHNIK is a member of the research staff at the Center for Communications Research, La Jolla. He is also the author of BibT_EX, a widely used bibliography processor.